Chapter 7

HIGH-PRECISION MHD SIMULATION

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7.1 Introduction

Since the middle of 1960's, studies of space plasma physics using computer simulation have started. At first, quite specialized and simplified one dimensional simulation models were used. From 1970's magnetohydrodynamic (MHD) simulation and particle simulation began to be used for various problems in plasma physics. Now, together with developments of "super computer ", such simulations show a great power in the study of solar-terrestrial plasma physics. In this lecture, a new scheme of MHD simulation as well as the traditional scheme and the estimation of the numerical preciseness of those schemes are introduced.

In MHD simulation, we solve the following equations (MHD equation)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{7.1}$$

$$\rho \frac{d\,\boldsymbol{v}}{dt} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p \tag{7.2}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} \tag{7.3}$$

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} + \eta \boldsymbol{j} \tag{7.4}$$

$$\boldsymbol{j} = \frac{1}{\mu_0} \nabla \times \boldsymbol{B} \tag{7.5}$$

$$\frac{\partial p}{\partial t} + \nabla \cdot (p\boldsymbol{v}) = (\gamma - 1)(-p\nabla \cdot \boldsymbol{v} + \eta \boldsymbol{j}^2) \tag{7.6}$$

where $\boldsymbol{B}$, $\boldsymbol{E}$, $\boldsymbol{j}$, $\boldsymbol{v}$, ρ, p, η, and γ denote the magnetic field, the electric field, the electric current, the plasma flow velocity, the plasma density, the plasma pressure, the electric resistance and the ratio of the specific heats, respectively.

Essentially, MHD simulation code is just a program to solve above time development equations under certain boundary conditions and initial conditions. Usually, we divide the simulation box (the physical region in which we want to solve above equations) into many grid points where the physical quantities are defined. And the derivatives of each quantity is given by the finite difference method.

In the following introduction of the schemes, for simplicity, one dimensional wave propagation problem is considered.

$$\frac{\partial u}{\partial t} = V_p \frac{\partial u}{\partial x} \tag{7.7}$$

where V_p is the phase velocity of the wave. Initially, the value of u at each grid point ($x = x_j = \Delta x \cdot (j-1), j = 1 \sim N_x : \Delta x$; grid interval) is given. Then, the problem is how to decide the value at $t = t_{n+1}$ from $t = t_n (t_n = n\Delta t : \Delta t$; time step).

7.2   2-Step Lax-Wendroff Scheme

Here, the 2-step Lax-Wendroff method is introduced. This scheme is very traditional one and has been used by many researchers of computational physics. In this scheme, intermediate values $u_{j+1/2}$ at the half timesteps $t_{n+1/2}$ and the half mesh point $x_{j+1/2}$ at the first step are defined. These are calculated by

$$u_{j+1/2}^{n+1/2} = \frac{u_{j+1}^n + u_j^n}{2} + \frac{V_p \Delta t}{2} \cdot \frac{u_{j+1}^n - u_j^n}{\Delta x} \tag{7.8}$$

In the second step, using these intermediate values, the updated value u_j^{n+1} is calculated by

$$u_j^{n+1} = u_j^n + V_p \Delta t \cdot \frac{u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}}{\Delta x} \tag{7.9}$$

Thus, we can get the time developing solution of the equation through these two procedures (this is why this scheme is called two-step Lax-Wendroff method).

In order to consider the stability of this scheme, or in other words, how large numerical damping the wave suffers by this scheme, we derive the amplification factor for one time step. We assume the wave which is proportional to $exp(ikx)$, then at the first step

$$u_{j+1/2}^{n+1/2} = u_j^n \cdot [\{exp(ik\Delta x) + 1\}/2 + \frac{\alpha}{2}\{exp(ik\Delta x) - 1\}] \tag{7.10}$$

$$u_{j-1/2}^{n+1/2} = u_j^n \cdot [\{1 + exp(-ik\Delta x)\}/2 + \frac{\alpha}{2}\{1 - exp(-ik\Delta x)\}] \tag{7.11}$$

where $\alpha = V_p \Delta t / \Delta x$. Then, at the second step,

$$
\begin{aligned}
u_j^{n+1} &= u_j^n \cdot [1 \; + \; i\alpha \sin(k\Delta x) + \alpha^2 \{\cos(k\Delta x) - 1\}] &\quad (7.12)\\
&= G \cdot u_j^n &\quad (7.13)
\end{aligned}
$$

Here, G is a numerical amplification or damping factor which is called "G-factor". The discussion about the numerical accuracy using G-factor will be done together with G-factor of the new high precision scheme.

7.3   High Precision Scheme

This scheme essentially adopts the 4th order Runge-Kutta-Gill method for time integration which has been well-known as the traditional integration method, but no one (maybe) has used this scheme for the MHD simulation. For the space derivatives, we use the usual centered finite difference method in the 4th order. This method consists of 4 steps as follows;

$$
d_{j,1} = \Delta t \cdot \frac{-(u_{j+2}^n - u_{j-2}^n) \; + \; 8(u_{j+1}^n \; - \; u_{j-1}^n)}{12\Delta x} \tag{7.14}
$$

$$
f_{j,1} = 0.5 \cdot d_{j,1} \tag{7.15}
$$

$$
q_{j,2} = d_{j,1} \tag{7.16}
$$

$$
d_{j,2} = \Delta t \cdot \frac{-(f_{j+2,1} - f_{j-2,1}) \; + \; 8(f_{j+1,1} \; - \; f_{j-1,1})}{12\Delta x} \tag{7.17}
$$

$$
f_{j,2} = (1 - \sqrt{0.5})(d_{j,2} - q_{j,2}) \tag{7.18}
$$

$$
q_{j,3} = 2(1 - \sqrt{0.5})d_{j,2} + 2(3\sqrt{0.5} - 2)q_{j,2} \tag{7.19}
$$

$$
d_{j,3} = \Delta t \cdot \frac{-(f_{j+2,2} - f_{j-2,2}) \; + \; 8(f_{j+1,2} \; - \; f_{j-1,2})}{12\Delta x} \tag{7.20}
$$

$$
f_{j,3} = (1 + \sqrt{0.5})(d_{j,3} - q_{j,3}) \tag{7.21}
$$

$$
q_{j,4} = 2(1 + \sqrt{0.5})d_{j,3} - (3\sqrt{0.5} + 2)q_{j,3} \tag{7.22}
$$

$$
d_{j,4} = \Delta t \cdot \frac{-(f_{j+2,3} - f_{j-2,3}) \; + \; 8(f_{j+1,3} \; - \; f_{j-1,3})}{12\Delta x} \tag{7.23}
$$

$$
u_j^{n+1} = d_{j,4}/6 - q_{j,4}/3 \tag{7.24}
$$

In this case, the amplification factor, G, is calculated in the same manner as in the Lax-Wendroff case so that

$$
G = 1 \; + \; H \; + \; H^2/2 \; + \; H^3/6 \; + \; H^4/24 \tag{7.25}
$$

$$
H = -i\frac{\alpha}{3}[4 - \cos(k\Delta x)]\sin(k\Delta x) \tag{7.26}
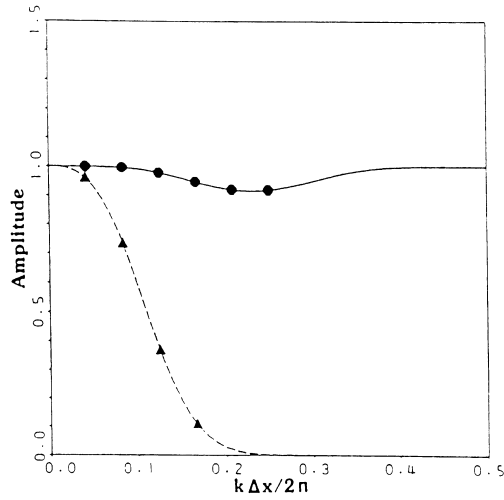$$

Figure 7.1: Ratio of the wave amplitude after 10 wave transit times to the initail amplitude. ($\alpha = 0.8$)
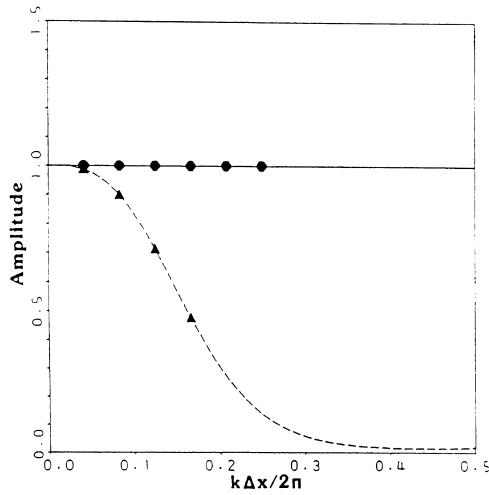

Figure 7.2: Ratio of the wave amplitude after 10 wave transit times to the initail amplitude. ($\alpha = 0.1$)

Figures 7.1 and 7.2 show the examples of the ratio of the wave amplitude after 10 wave transit times to the initial amplitude, against the normalized wave number $k\Delta x/2\pi$, when $\alpha = 0.8$ (Figure 7.1) and $\alpha = 0.1$ (Figure 7.2). The solid and dashed lines express the theoretical curves of the high-precision scheme and Lax-Wendroff scheme, respectively, while the solid circles and triangles are the respective simulation results. As is clearly seen in both figures, waves of short wavelength are much more strongly damped in the Lax-Wendroff scheme than in the high-precision scheme. For example, when $\alpha = 0.8$ in Figure 7.1, the amplitude of the wave of wavelength $12\Delta x$ came down to 73% of the initial amplitude in the Lax-Wendroff scheme, while the wave maintained its initial amplitude (99.6%) in the high-precision scheme. Furthermore, in the case of the wavelength $6\Delta x$, only 11% of the initial amplitude survived in the Lax-Wendroff scheme, while 95% of the wave amplitude survived in the high precision scheme.

Suppose that we make a criterion that a wave will be physical when the damping is less than 10% at 10 wave transit times. Then, only the wave with wavelength longer than $17\Delta x$ satisfies the criterion in the Lax-Wendroff scheme, while all wavelength waves can satisfy this criterion in the high precision scheme(the largest damping is 8%). It should be noticed that when we execute global simulation of the solar wind - magnetosphere interaction, Δx is usually chosen as large as 0.5 $R_E(R_E$; the earth radius) and the wavelength $17\Delta x$ corresponds to about $9R_E$. When we consider the case where $\alpha = 0.1$, the difference between the two schemes is more obvious. This means that most of the phenomena in the global interactions with characteristic change times, say, a few wave transit times or larger, may suffer a serious damping in the 2-step Lax-Wendroff method. Thus, even though some qualitative resemblances to observations are obtained by the simulations using Lax-Wendroff method, quantitative arguments are highly irrelevant.

Figure 7.3 shows one of the examples of the ratio of the wave amplitude after 10 wave transit times to the initial amplitude, as a function of the normalized time step α when $k\Delta x/2\pi = 0.125$. The solid and dashed lines express the theoretical curves of the high precision scheme and the Lax-Wendroff scheme, respectively, while the solid circles and triangles are the respective simulation results. One can easily see that the 2-step Lax-Wendroff scheme becomes numerically unsatble when $\alpha > 1$. Therefore, when we wish to make a long simulation run with the Lax-Wendroff method, we meet a conflict: Namely, when α is chosen close to 1, the numerical scheme is easily unstable against the small numerical fluctuation of the density and/or the magnetic field because the Alfven velocity determines the characteristics of the system in the actual MHD simulation. Thus, large artificial damping terms must be added in the equations which make numerical artifacts. In contrast, when α is chosen much less than 1, it takes a large number of time steps, and thus a long computer time is required to make a long simulation run. When the intermediate α is chosen, the wave amplitude suffers a serious numerical damping. In contrast with the Lax-
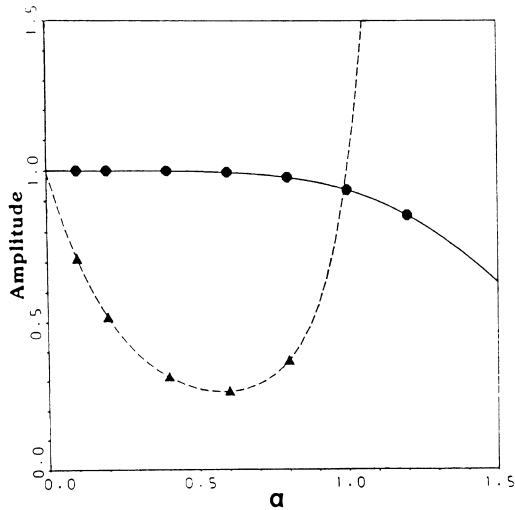
Figure 7.3: Ratio of the wave amplitude after 10 wave transit times to the initail amplitude as a function of α.

Wendroff scheme, the high-precision scheme is stable even for $\alpha > 1$, so that a physically sound, long time scale simulation can become real.

This high precision scheme is found to be numerically unstable when $\alpha > 2.2$ for any wave number. In reality, the phase error of the propagating wave is more severe than the amplitude error. In fact, the phase condition of the propagating wave shows that α should be chosen to be less than 1.2 for the physically sound use of this new scheme.

7.4   Other Schemes

There are a few more schemes which have good accuracy in space and/or time to some extent, for example, the upstream scheme which is the space differential scheme. This scheme was developed for the shock problem in fluid dynamics. In this scheme, It is considered that the information is carried by the fluid itself (fluid movement), and that the quantity on the upstream side has larger influence on the space differential than that on the downstream when the finite difference is taken. In plasma physics, however, the information is usually carried by the wave in plasma such as Alfven wave or magnetosonic wave. Therefore, the upstream scheme is not so good scheme for the problems in plasma physics (of course, to the shock problem in plasma physics such as the bow shock, this

scheme may make a good contribution). Some other schemes which are developed in fluid dynamics, are not applicable to the plasma physics by the other reason. Namely, they require more boundary conditions outside the simulation box, while boundary conditions in plasma physics are so complicated and it is almost impossible to give the boundary conditions outside the simulation box. From these viewpoints, we recommend the MHD code which adopts above introduced high-precision scheme with the 4th order centered finite difference method in space for the general use in plasma physics (All the MHD codes of our Simulation Center adopt this scheme and make quantitative simulations in fusion plasma physics).