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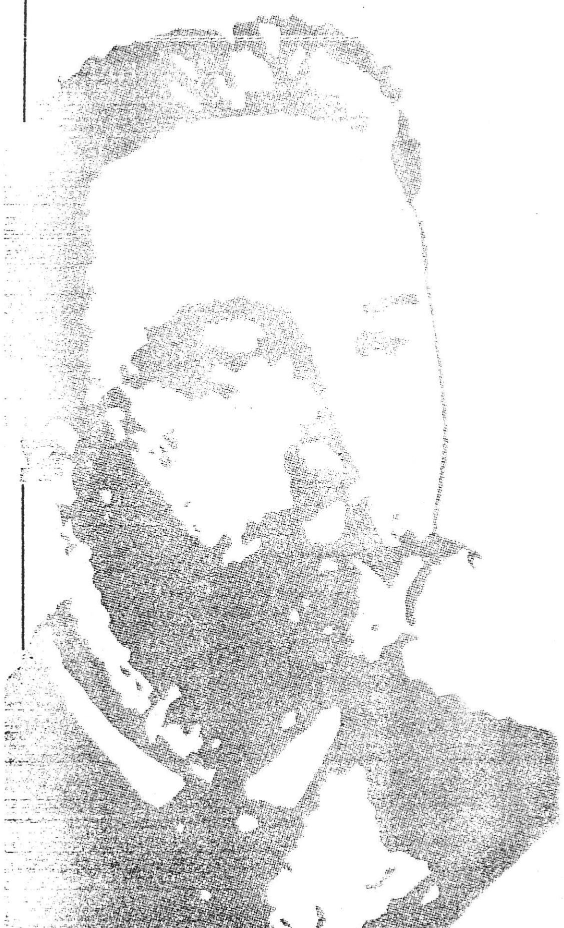

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ON THE TURBULENCE OF WHISTLERS

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The weark turbulence of whistlers propagating at large angles to the external magnetic field is considered in the case where decays are the basic nonlinear interactions. Anisotropic Kolmogorov spectra are found and their locality is investigated.

Whistlers (helicones or spiral waves) propagate in the collisionless magnetoactive plasma when $\omega_{Hi} \ll \omega \ll \omega_{He}$. Under an additional assumption that $\omega_{pe}^2 \gg k^2c^2$. the whistler spectrum has the form

$$\omega = k \mid k_z \mid c^2\omega_{He}/\omega_{pe}^2.$$

In the isothermal plasma, the principal nonlinear interactions determining the weak (wave) turbulence of whistlers are three-plasmon decays

$$W = W_1 + W_2, \tag{1}$$

where W stands for the plasmon energy, and induced scatterings of whistlers by ions and electrons. The Kolmogorov turbulence spectra (i.e. power-law stationary spectra determined by constant flows in the k-space of conserved quantities) can only exist in the limiting cases of whistler propagation at either large or small angles to the external magnetic field. In these cases the theory becomes scale invariant with respect to either of the arguments $k_\perp$ and k_z separately, which allows seeking stationary spectra in the form $N_k \sim \mid k_z \mid^\alpha k_\perp^\beta$. In the limit of small angles $\cos^2\theta \approx 1$ the decays are inhibited by the laws of conservation of energy and momentum, and the principal nonlinear interactions are scatterings on particles leading to the stationary turbulence spectrum $\mathcal{E}_\omega \sim \omega^{-1/2}$ [1]. In the opposite case, that is, for large propagation angles $\cos^2\theta \ll 1$, the predominant processes are decays. It will be shown in the sequel that in this limit there also exist power-law stationary spectra.

In the decay interaction (1), conserved quantities are energy integrals and z-components of wave momentum. The most interesting is the case where the latter integral is other than zero, and we therefore assume that the predominant wave propagation proceeds into one half space $k_z > 0$.

The balance equation for the number of plasmons N_k in the decay interaction has the form

$$
\begin{aligned}
\dot{N}_{\mathbf{k}} = & -\int d^3k_1 d^3k_2 w(\mathbf{k}\mathbf{k}_1\mathbf{k}_2)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\delta(\omega-\omega_1-\omega_2)\times \\
& \times(NN_1 + NN_2 - N_1N_2)+ \\
& +\int d^3k_1 d^3k_2 w(\mathbf{k}_1\mathbf{k}\mathbf{k}_2)\delta(\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2)\delta(\omega_1-\omega-\omega_2)\times \\
& \times(NN_1 + N_1N_2 - NN_2)+ \\
& +\int d^3k_1 d^3k_2 w(\mathbf{k}_2\mathbf{k}_1\mathbf{k})\delta(\mathbf{k}_2-\mathbf{k}_1-\mathbf{k})\delta(\omega_2-\omega_1-\omega)\times \\
& \times(N_1N_2 - NN_2 - N_1N),
\end{aligned}
\tag{2}
$$

where $N \equiv N_{\mathbf{k}}, N_1 \equiv N_{\mathbf{k}_1}, N_2 \equiv N_{\mathbf{k}_2}$. To calculate the decay probability $w(\mathbf{k}\mathbf{k}_1\mathbf{k}_2)$ it is necessary to know the nonlinear response of the second order $S_{ijk}(\mathbf{k}\mathbf{k}_1\mathbf{k}_2)$ determined by the expression

$$
\begin{aligned}
j_i^N(\omega,\mathbf{k}) = & \, 2\int d^3k_1 d^3k_2 S_{ijk}(\mathbf{k}\mathbf{k}_1\mathbf{k}_2)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\delta(\omega-\omega_1-\omega_2)\times \\
& \times E_j(\omega_1,\mathbf{k}_1)E_k(\omega_2,\mathbf{k}_2).
\end{aligned}
\tag{3}
$$

We shall assume plasma particles to be magnetized: $k_\perp V_{T\alpha} \ll \omega_{H\alpha}$ and $\omega \gg k_z V_{T\alpha}$. Then the nonlinear current (3) can be calculated on the basis of the equations of cold magnetic hydrodynamics for electrons, which yields the following expression for the decay probability

$$
\begin{aligned}
w = & \, F\frac{k_{1z}k_{2z}}{k_z k_\perp k_{1\perp}k_{2\perp}}(k_\perp + k_{1\perp} + k_{2\perp})^2(k_{1\perp} - k_{2\perp})^2\times \\
& \times \Delta^2(k_\perp, k_{1\perp}, k_{2\perp}),
\end{aligned}
\tag{4}
$$

where $F = (2\pi)^4 c^4 \omega_{He}^2/4n_0 m_e \omega_{He}\omega_{pe}^4$; Δ is the area of the triangle constructed on the vectors $\mathbf{k}_\perp, \mathbf{k}_{1\perp}, \mathbf{k}_{2\perp}$. Detailed calculations of the second-order response are presented in ref. [2]. The expression (4) is written straightforwardly in the limit of large angles, and the general expression is restored by the substitution $k_\perp \to k, k_{1\perp} \to k_1, k_{2\perp} \to k_2$.

The terms of the expression (2) will each be integrated over the angles in the plane perpendicular to the magnetic field, and the stationary spectrum will be sought in the form $N_k = Ak_z^\alpha k_\perp^\beta$, $k_z > 0$. Substituting this expression in (2) and making Zakharov – Kats – Kontorovich transformation [3, 4] $k_{1z,\perp} \to k_{z,\perp}^2/k_{1z,\perp}, k_{2z,\perp} \to k_{z,\perp}k_{2z,\perp}/k_{1z,\perp}$ in the second term of (2) and a similar transformation $k_{2z,\perp} \to k_{z,\perp}^2/k_{2z,\perp}, k_{1z,\perp} \to k_{z,\perp}k_{1z,\perp}/k_{2z,\perp}$ in the third term, we reduce the collision integral in (2) to the form

$$
\int dk_{1z}dk_{2z}dk_{1\perp}dk_{2\perp}k_{1\perp}k_{2\perp}w(kk_1k_2)\delta(k_z - k_{1z} - k_{2z})\delta(\omega-\omega_1-\omega_2)\times
$$

$$\times A^2[1 - (k_z/k_{1z})^{2+2\alpha}(k_\perp/k_{1\perp})^{8+2\beta} - (k_z/k_{2z})^{2+2\alpha}(k_\perp/k_{2\perp})^{8+2\beta}]\times \tag{5}$$
$$\times[(k_z k_{1z})^\alpha(k_\perp k_{1\perp})^\beta + (k_z k_{2z})^\alpha(k_\perp k_{2\perp})^\beta + (k_{1z}k_{2z})^\alpha(k_{1\perp}k_{2\perp})^\beta].$$

The stationary spectra are found from the condition of equality to zero of the collision integral (5). It can be readily verified that the second square bracket in (5) is nullified by the solutions 1) $\alpha = -1$, $\beta = -1$, 2) $\alpha = -1$, $\beta = 0$, and the first square bracket by the solutions 3) $\alpha = -3/2$, $\beta = -9/2$ and 4) $\alpha = -3/2, \beta = -4$. The first two solutions correspond to the limiting cases of equilibrium distribution $N_k \sim 1/(\omega + ak_z)$, nullify identically each of the three terms in (2) and correspond to zero fluxes of energy and momentum in the k-space. Simple dimensional estimations show that the third solution must correspond to a constant energy flux, and the fourth to a constant flux of the z-component of momentum in the k-space. However, these solutions may exist only providing the turbulence is local, that is, the original collision integral in (2) is convergent on the spectra corresponding to solutions 3 and 4. Convergences may occur as $k_1 \to 0$ and $k_1 \to \infty$. The convergence conditions for $k_1 \to \infty$ impose weak restrictions upon α and β, which are satisfied for solutions 3 and 4. Let us consider in more detail the most dangerous limit $k_1 \to 0$. Divergences here are due to the first and third terms of (2). We introduce the notation $k_{1z} \equiv p_1$, $k_{1\perp} \equiv q_1$. When $p_1 \to 0$ and $q_1 \to 0$, by virtue of the existence of two δ-functions we have

$$I_{st} \sim -\int_0\int_0 q^3 p_1 p^{-1}\Delta N_1[(2N - N(p - p_1, q + qp_1/p) -$$
$$- N(p + p_1, q - qp_1/p)) + 7p^{-1}p_1(N - N(p - p_1, q + qp_1/p))]dq_1 dp_1,$$
$$\Delta = \frac{1}{2}qq_1(1 - q^2p_1^2/q_1^2p^2)^{1/2}.$$

The intergation is carried out over the region $0 \le p_1/p \le q_1/q$. The condition of convergence of this integral for $k_1 \to 0$ is this: $\alpha + \beta > -6$, $\alpha > -4$. So, the spectrum 3 is nonlocal the collision integral diverges logarithmically, while the spectrum 4 corresponds to the local flux of the z-component of momentum. The components of the vector of the flux of the z-component of momentum are expressed in terms of the derivatives of the collision integral with respect to the variables α and β at the point $\alpha = -3/2$, $\beta = -4$ as follows [3]: $P_z = q^{-2}\partial I/\partial\alpha$, $P_\perp = p^{-1}q^{-1}\partial I/\partial\beta$, where I is the dimensionless collision integral (5). Numerical estimations yield $(\partial I/\partial\alpha)/(\partial I/\partial\beta) \simeq 3,13$, $\partial I/\partial\alpha > 0$, $\partial I/\partial\beta > 0$. The flux of the z-component of momentum is aligned towards large k.

Concluding we note that the plasma isothermicity condition assumed above is important because in the case $T_e \gg T_i$ an ion sound may exist in the region $\omega_{Hi} \ll \omega \ll \omega_{pi}$, and the decisive role in the formation of the spectrum may be played by decay interactions between whistlers an sound and by the nonlinear whistler transformation into sound on plasma particles. The probabilities of such processes and estimates of possible spectra were obtained in ref. [5]. Our consideration may also appear to be inapplicable if whistler frequencies get into the range of longitudinal waves which have the following dispersion [6] (for large angles)

$$\omega^2 = \frac{\omega_{pe}^2 \omega_{He}^2}{\omega_{pe}^2 + \omega_{He}^2} \cos^2\theta$$

when $\cos^2\theta \ll m_e/m_i$ and

$$\omega^2 \approx \omega_{Hi}\omega_{He}$$

when $\cos^2\theta \le m_e/m_i$. In this case, a whistler may decay into a whistler and a longitudinal wave, and from the conservation laws it follows that it is $k_\perp$ rather than k_z of the whistler that undergoes appreciable changes (we assume the magnetic field to be not very strong: $\omega_{He} \ll \omega_{pe}^3/k^2c^2$). Such processes would result in a rapid energy transfer over $k_\perp$ and a slow transfer over k_z. The opposite process, i.e. the merging of two whistlers into a longitudinal wave is impossible because the conservation laws require in this case that the whistlers have k_z of opposite signs while we believe that there are no whistlers with $k_z < 0$.

Our consideration holds true for decays with participation of the given longitudinal waves forbidden, that is, for whistler frequencies lower than the longitudinal wave frequencies: $\omega^2 < \omega_{He}\omega_{Hi}$, which leads to a weak restriction on wave vectors and propagation angles of whistlers:

$$k^2c^2\cos\theta/\omega_{pe}^2 \le (m_e/m_i)^{1/2}.$$

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