Lecture 2: Two-body problem (5 Sep 14)

Effective Monday Sept 8: class meets in 2116 Chamberlin

A. Relative motion of two bodies, FW Sec. 4

1. $\mathbf{R} \equiv \vec{R}$ [bold face for vectors]; dot for time derivative, e.g., $\dot{\mathbf{R}}$.

2. Formulate Newton's laws using two bodies with (internal) central forces – second and third laws: ($\mathbf{F}_{21}$ is force of 2 on 1)

$$d\mathbf{p}_1/dt = \mathbf{F}_{21}; \mathbf{F}_{21} = -\mathbf{F}_{12}$$

$\mathbf{F}_{21} = [-d\Phi/dr](\mathbf{r}_1 - \mathbf{r_2})/r_{12} = [-d\Phi/dr]\hat{r}_{12}$. Check $\nabla_1 \times \mathbf{F} = 0$, either using $\mathbf{F} = -\nabla_1 \Phi$ or with $\mathbf{F} = G(r)\mathbf{r}$ and using Cartesians.

3. Center-of-mass and relative coordinates defined for masses m_1 and m_2, $M = m_1 + m_2$: (L & L, Sec. 13); and inverse

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}; \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}; \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}$$

4. The corresponding relations for center-of-mass momentum and relative momentum are (quantum mechanics: commutators check)

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2; \mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{M} = \frac{m_1 m_2}{M}(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2)$$

$$\mathbf{p}_1 = \frac{m_1}{M}\mathbf{P} + \mathbf{p}; \mathbf{p}_2 = \frac{m_2}{M}\mathbf{P} - \mathbf{p}$$

5. Mechanical momenta $\mathbf{p}_j = m_j\dot{\mathbf{r}}_j$, the relative momentum has reduced mass μ:

$$\mathbf{p} = \mu\dot{\mathbf{r}}; \mu = \frac{m_1 m_2}{m_1 + m_2}; \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

6. No external forces: conservation of total (center-of-mass) momentum

$$\frac{d}{dt}\mathbf{P} = \mathbf{F}_{21} + \mathbf{F}_{12} = 0$$

$\Rightarrow$ explicit solution for $\mathbf{R}(t)$ using $\mathbf{V}$ = constant center-of-mass velocity.

$$0 = \frac{d}{dt}\mathbf{P} = M\frac{d^2}{dt^2}\mathbf{R} \Rightarrow \mathbf{R} = \mathbf{R}_0 + \mathbf{V}t$$

7. The total angular momentum $\mathbf{L}$ and relative angular momentum $\mathbf{l}$ are

$$\mathbf{L} = \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2 = \mathbf{R} \times \mathbf{P} + \mathbf{r} \times \mathbf{p}\,;\mathbf{l} \equiv \mathbf{r} \times \mathbf{p}$$

$$\text{no external forces} \Rightarrow \frac{\mathrm{d}}{\mathrm{dt}}[\mathbf{R} \times \mathbf{P}] = 0$$

8. Time derivatives of the relative momentum and angular momentum:

$$\frac{d}{dt}\mathbf{p} = \frac{m_2}{M}\mathbf{F}_{21} - \frac{m_1}{M}\mathbf{F}_{12} = \mathbf{F}_{21};\, \frac{d}{dt}\mathbf{l} = \mathbf{r} \times \mathbf{F}_{21}$$

9. Central force $\mathbf{F} = F\mathbf{r}/r \Rightarrow \mathbf{l}$ is a constant of the motion, i.e., conserved:

$$\frac{d}{dt}\mathbf{l} = 0$$

$\mathbf{r} \cdot \mathbf{l} = \mathbf{r} \cdot (\mathbf{r} \times \mathbf{p}) = 0$; therefore the relative motion remains in a plane perpendicular to $\mathbf{l}$, i.e., motion in 2D.

10. Assume (central) force derived from a potential energy function:

$$\mathbf{F}_{21} = -\frac{d\Phi}{dr}\frac{\mathbf{r}}{r} = -\vec{\nabla}_1\Phi(|\mathbf{r}_1 - \mathbf{r}_2|)$$

The total kinetic plus potential energy is

$$E = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + \Phi(r) = \frac{\mathbf{p}^2}{2\mu} + \frac{\mathbf{P}^2}{2M} + \Phi(r)$$

The center-of-mass kinetic energy is constant; the time-derivative of the remaining terms shows E = constant.

$$\frac{dE}{dt} = \frac{1}{\mu}\mathbf{p} \cdot \dot{\mathbf{p}} + \nabla\Phi \cdot \dot{\mathbf{r}} = \dot{\mathbf{r}} \cdot \dot{\mathbf{p}} - \mathbf{F} \cdot \dot{\mathbf{r}} = 0$$

B. Effective potential, FW 3, LL 14

1. The two-body problem with conservative internal central forces has reduced to a relative motion problem with

$$\mathbf{l} = \text{constant}; \mathrm{E} = \frac{\mathbf{p}^2}{2\mu} + \Phi(\mathrm{r}) = \text{constant}$$

2. Introduce plane polar coordinates r, ϕ in the plane perpendicular to $\mathbf{l}$.

$$x = r\cos\phi; y = r\sin\phi; \dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi; \dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$$

$$\mathbf{l} = \hat{z}[xp_y - yp_x] = \mu r[c_\phi\dot{y} - s_\phi\dot{x}]\hat{z} = \mu r^2\dot{\phi}\hat{z} \equiv \ell\hat{z}$$

   The kinetic energy is, using the angular momentum

$$K = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\phi}^2) = \frac{\mu}{2}\dot{r}^2 + \frac{\ell^2}{2\mu r^2}$$

   The constant energy relation is expressed as

$$E = \frac{\mu}{2}\dot{r}^2 + \Phi_{eff} = \text{constant}; \Phi_{\text{eff}} = \Phi(\mathrm{r}) + \frac{\ell^2}{2\mu\mathrm{r}^2}$$

3. The original problem has 6 vector components $\mathbf{r}_1, \mathbf{r}_2$. The conservation laws have reduced the calculation of dynamics to the solution of

$$\dot{r}^2 = (2/\mu)[E - \Phi_{eff}(r)] \rightarrow \frac{dr}{\sqrt{(2/\mu)[E - \Phi_{eff}(r)]}} = \pm dt$$

   which "can be integrated" to give $r(t)$. Then $\dot{\phi} = \ell/[\mu r(t)^2] \rightarrow \phi(t)$.

C. Kepler problem FW pg. 13, LL 15

1. Explicit solutions for $\Phi \equiv -\Gamma/r$ (factor differs from FW by m_1). $\Gamma \equiv Gm_1m_2$ for Newtonian gravity.

2. The problem has been reduced to solving for "one degree of freedom." This is not enough to guarantee closed orbits [where the ratio of periods of ϕ and r is a rational number, Goldstein Sec. 3.6, homework P1.12]. The Kepler problem does have closed orbits, though. First, use a brute-force solution to integrate the $\phi - r$ relation obtained by re-writing

$$\dot{r} = (dr/d\phi)d\phi/dt = (\ell/\mu r^2)(dr/d\phi)$$

   and using

$$\dot{r} = \pm\sqrt{(2/\mu)(E - \Phi_{eff})}$$

3. The calculus goes easier with $u \equiv 1/r$, $\dot{r} = -(\ell/\mu)du/d\phi$

$$du/d\phi = \mp\sqrt{(2\mu E/\ell^2) - u^2 + u(2\mu\Gamma/\ell^2)}$$

4. Then the indefinite integral is

$$\phi - \phi_0 = \mp\int_{u_0}^{u} du/\sqrt{(2\mu E/\ell^2) - u^2 + u(2\mu\Gamma/\ell^2)}$$

5. Do the integral: (1) define $v = u - (\mu\Gamma/\ell^2)$ to "complete the square," (2) make a trigonometric substitution $v = \alpha\cos\psi$ with $(\alpha \geq 0)$

$$\alpha = \sqrt{(2\mu E/\ell^2) + (\mu\Gamma/\ell^2)^2}$$

   Then the indefinite integral takes the form (absorb an integration constant in the difference between ϕ_0 and $\tilde{\phi}$)

$$\cos^{-1}(v/\alpha) = \mp(\phi - \tilde{\phi})$$

$$\Rightarrow \frac{1}{r} = \frac{\mu\Gamma}{\ell^2}[1 + \epsilon\cos(\phi - \tilde{\phi})] = \frac{\mu\Gamma}{\ell^2}[1 - \epsilon\cos(\phi)]$$

   after choosing an origin for ϕ and eccentricity defined by

$$\epsilon = \frac{\ell^2}{\mu\Gamma}\sqrt{(2\mu E/\ell^2) + (\mu\Gamma/\ell^2)^2} = \sqrt{1 + (2E\ell^2/\mu\Gamma^2)}$$

HW for September 8

1. #1.4. Rocket problem for static (no rotation) earth. Treat as 1D motion, along $\hat{z}$-axis.

2. #1.5 First sketch the potentials to get sense of the energy ranges for (periodic) bounded motion. Then set up the integral $\tau = \int dx/v_x$ and try to extract the functional dependence on the turning point and energy.

3. #1.6 Start from the vector equation of motion written in Cartesians [$x = r\cos\phi, y = r\sin\phi$] and then examine how to extract the conservation of energy and angular momentum from these equations.