Lecture 27: Hamiltonian II (3 Nov 14)

A. Hamilton equations

1. In Lagrangian formulation the independent variables are (generalized) coordinates q and velocities $\dot{q}$. In the Hamiltonian formulation, they are (generalized) coordinates q and canonical momenta p. The transformation from L to H is a Legendre transformation and the independent variables change from $q_,, \dot{q}_j$ to q_j, p_j (i.e., the partials have different "other" coordinates held fixed for H and L)

2. If the Lagrangian has no explicit time dependence) $L = L(\{q_j\}, \{\dot{q}_j\})$, $H \equiv \sum_j \dot{q}_j p_j - L$ is conserved/constant (usually, the energy): $dH/dt = 0$.

3. The Lagrange equations of motion are
$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{d}{dt}p_j;\ p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$$
while the Hamilton equations are
$$\frac{\partial H}{\partial p_j} = \dot{q}_j\,;\ \frac{\partial H}{\partial q_j} = -\dot{p}_j$$
and follow from
$$dH = \sum_j [p_j d\dot{q}_j + \dot{q}_j dp_j] - \sum_j [\frac{\partial L}{\partial q_j}dq_j + \frac{\partial L}{\partial \dot{q}_j}d\dot{q}_j] - \frac{\partial L}{\partial t}dt$$
$$= \sum_j [\dot{q}_j dp_j - \frac{\partial L}{\partial q_j}dq_j] - \frac{\partial L}{\partial t} = \sum_j [\dot{q}_j dp_j - \dot{p}_j dq_j] + \frac{\partial H}{\partial t}dt$$

4. Hamilton's principle with q_j and p_j as independent variables:
$$\delta \int_{t_1}^{t_2} L dt = 0 \rightarrow \delta \int_{t_1}^{t_2} [\sum_j \dot{q}_j p_j - H] dt = 0$$

5. For a conservative system with $\Phi = \Phi(\{q_j\})$ and $x_j = x_j(\{q_\sigma\})$, holonomic *time-independent* constraints, (using σ as a notation that the number of spatial variables may be less than $3N$)
$$\sum_\sigma p_\sigma \dot{q}_\sigma = 2T \Rightarrow H = T + \Phi$$

B. Liouville theorem

1. G Sec. 9.9; LL Sec. 46; Symon Sec.9.11; Kardar (Stat Phys) Sec. 3.4; Halley (Stat Mech) Chap.1

2. Phase space $\{q_j, p_j\}$. Consider a differential element in the "j-th" piece $dq_j\, dp_j$ and the element after time δt (differential displacement of the ends): $dq_j \to dq_j + [\partial \dot{q}_j/\partial q_j] dq_j\, \delta t$, $dp_j \to dp_j + [\partial \dot{p}_j/\partial p_j] dp_j\, \delta t$. Hence

$$dq_j\, dp_j \to dq_j\, dp_j [1 + (\delta t)\{\partial \dot{q}_j/\partial q_j + \partial \dot{p}_j/\partial p_j\}]$$

   The coefficient of δt is then

$$\{\partial \dot{q}_j/\partial q_j + \partial \dot{p}_j/\partial p_j\} = \frac{\partial}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial}{\partial p_j}\frac{\partial H}{\partial q_j} = 0$$

   The volume element remains the same size as it follows system obeying Hamilton equations. The phase space density behaves like an incompressible fluid.

3. Formulate the equation of continuity in phase space:

$$\partial \rho/\partial t + \vec{\nabla} \cdot \rho \mathbf{v} = 0$$

   a flow with generalized coordinates and canonical momenta $\{q_j, p_j\}$; $\mathbf{v} = \{\dot{q}_j, \dot{p}_j\}$ and a divergence:

$$\vec{\nabla} \cdot \mathbf{v} = \sum_{j=1}^{f} [\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j}] = \sum_{j=1}^{f} \frac{\partial^2 H}{\partial q_j \partial p_j} - \frac{\partial^2 H}{\partial p_j \partial q_j} = 0$$

4. Then $d\rho/dt \equiv \partial \rho/\partial t + \mathbf{v} \cdot \vec{\nabla} \rho = 0$, the moving volume element does not change size.

C. Canonical transformations

1. FW Secs. 34,35; LL Secs. 46, 47; G Sec. 9.1.

2. Examine transformations to new coordinates Q_j, P_j

$$p_j = p_j(P_i, Q_i, t);\ q_j = q_j(P_i, Q_i, t)$$

that keep the form of Hamilton's equations for the new Hamiltonian $\tilde{H}$:

$$\dot{Q}_j = \frac{\partial \tilde{H}}{\partial P_j};\ \dot{P}_j = -\frac{\partial \tilde{H}}{\partial Q_j}$$

3. This will be the situation if the modified Hamilton's principle is available for the new coordinates. That allows a bit of generality (the total time derivative gives an additive constant in the Hamilton's principle):

$$\sum_j \dot{q}_j p_j - H = \sum_j \dot{Q}_j P_j - \tilde{H} + \frac{dF_1(q, Q, t)}{dt}$$

$$\Rightarrow p_j = \frac{\partial F_1}{\partial q_j};\ P_j = -\frac{\partial F_1}{\partial Q_j};$$

$$\tilde{H}(Q, P, t) = H(p, q, t) + \frac{\partial F_1}{\partial t}$$

4. For Hamilton-Jacobi theory, one changes the variables in the added function F to q, P by the Legendre transformation

$$F_2 = \sum_j Q_j P_j + F_1$$

$$\sum_j \dot{q}_j p_j - H = \sum_j \dot{Q}_j P_j - \tilde{H} + \frac{dF_1(q, Q, t)}{dt} = -\sum_j Q_j \dot{P}_j - \tilde{H} + \frac{dF_2(q, P, t)}{dt}$$

Then set up the total time derivative of F_2:

$$\frac{dF_2}{dt} = \sum_j \frac{\partial F_2}{\partial q_j}\dot{q}_j + \frac{\partial F_2}{\partial P_j}\dot{P}_j + \frac{\partial F_2}{\partial t}$$

$$\Rightarrow p_j = \frac{\partial F_2}{\partial q_j};\ Q_j = \frac{\partial F_2}{\partial P_j}; \tilde{H}(Q, P, t) = H(p, q, t) + \frac{\partial F_2}{\partial t}$$

D. Hamilton-Jacobi theory, FW Sec. 35

1. Assume $S \equiv F_2$ is chosen to make $\tilde{H} = 0$. Then the dynamics problem has been "solved" because the equations of motion give

$$\dot{Q}_j = 0; \dot{P}_j = 0$$

2. The problem has been converted to finding the function S that satisfies:

$$H(\frac{\partial S}{\partial q_j}, q_j, t) + \frac{\partial S}{\partial t} = 0$$

3. Show S is the action by [with constant P_j]

$$\frac{dS}{dt} = \sum_j \frac{\partial S}{\partial q_j}\dot{q}_j + \frac{\partial S}{\partial t} = \sum_j \dot{q}_j p_j - H = L$$

4. For the case that H has no explicit time dependence, H is a constant of the motion and so can separate the time dependence of S as [W is Hamilton's characteristic function]

$$S(q_j, \alpha_j, t) = W(q_j, \alpha_j) - \alpha_1 t\,;\; H(\frac{\partial W}{\partial q_j}, q_j) = \alpha_1$$

5. Approximation to Schrödinger equation of quantum mechanics, with $p_j = (\hbar/\imath)\partial/\partial q_j$

$$H(\{p_j, q_j\})\Psi = \imath\hbar\partial\Psi/\partial t$$

$$\Psi \propto \exp(\imath S/\hbar)$$

and just keep the first terms $\partial S/\partial q_j$

$$[H(\{\partial S/\partial q_j, q_j\}, t) + \partial S/\partial t]\Psi = 0$$