Lecture 3: Kepler I (8 Sep 14)

Effective Monday Sept 8: class meets in 2116 Chamberlin

A. Review: 2-body relative motion

1. Two-body problem: relative motion with reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$, momentum $\mathbf{p} = \mu\dot{\mathbf{r}}$, angular momentum $\mathbf{l} = \mathbf{r}\times\mathbf{p}$. Conserved quantities with conservative central forces: $\mathbf{l}$ and energy

$$E = \frac{\mathbf{p}^2}{2\mu} + \Phi_{eff} = \frac{\mu}{2}\dot{r}^2 + \Phi_{eff}$$

   with an effective potential $\Phi_{eff} = \Phi(r) + \frac{\ell^2}{2\mu r^2}$. The problem has been reduced to solving for "one degree of freedom."

2. Motion remains in a plane perpendicular to $\mathbf{l}$ and a sketch of $\Phi_{eff}(r)$ shows the range of r for given E. The turning points r_1, r_2 of bounded motion can be set "by inspection" and the period τ evaluated by

$$\tau = 2\int_{r_1}^{r_2} dr/\sqrt{(2/\mu)[E - \Phi_{eff}]}$$

3. Explicit solutions for $\Phi \equiv -\Gamma/r$. $\Gamma \equiv Gm_1m_2$ for Newtonian gravity.

4. Brute-force solution for $r(\phi)$: transform $\dot{r} = \pm\sqrt{(2/\mu)(E - \Phi_{eff})}$ using

$$\dot{r} = (dr/d\phi)d\phi/dt = (\ell/\mu r^2)(dr/d\phi)$$

   and $u \equiv 1/r$, $\dot{r} = -(\ell/\mu)du/d\phi$

$$\Rightarrow du/d\phi = \mp\sqrt{(2\mu E/\ell^2) - u^2 + u(2\mu\Gamma/\ell^2)}$$

5. Then the indefinite integral is

$$\phi - \phi_0 = \mp\int_{u_0}^{u} du/\sqrt{(2\mu E/\ell^2) - u^2 + u(2\mu\Gamma/\ell^2)}$$

6. Do the integral: (1) define $v = u - (\mu\Gamma/\ell^2)$ to "complete the square," (2) make a trigonometric substitution $v = \alpha \cos\psi$ with $(\alpha \geq 0)$

$$\alpha = \sqrt{(2\mu E/\ell^2) + (\mu\Gamma/\ell^2)^2}$$

Then absorbing an integration constant in $\tilde{\phi}$

$$\cos^{-1}(v/\alpha) = \mp(\phi - \tilde{\phi})$$

$$\Rightarrow \frac{1}{r} = \frac{\mu\Gamma}{\ell^2}[1 + \epsilon\cos(\phi - \tilde{\phi})] = \frac{\mu\Gamma}{\ell^2}[1 - \epsilon\cos(\phi)]$$

after choosing an origin for ϕ and eccentricity defined by

$$\epsilon = \frac{\ell^2}{\mu\Gamma}\sqrt{(2\mu E/\ell^2) + (\mu\Gamma/\ell^2)^2} = \sqrt{1 + (2E\ell^2/\mu\Gamma^2)}$$

7. Circular orbit condition from $\epsilon = 0$ is

$$E = -(\mu\Gamma^2)/2\ell^2$$

Alternative: go to Φ_{eff} and construct force balance

$$\frac{d\Phi_{eff}}{dr} = 0 = \frac{\ell^2}{\mu r^3} - \frac{\Gamma}{r^2} \Rightarrow r_{eq} = \frac{\ell^2}{\mu\Gamma},\ \Phi_{eff}(r_{eq}) = -\frac{\mu\Gamma^2}{2\ell^2} = E$$

8. Check the solution, using $\dot{r} = (\ell/\mu r^2)dr/d\phi$, by:

$$E = \frac{\mu\dot{r}^2}{2} + \frac{\ell^2}{2\mu r^2} - \frac{\Gamma}{r} = \frac{\ell^2}{2\mu}[\{(1/r^2)dr/d\phi\}^2 + (1/r^2)] - \frac{\Gamma}{r}$$

and define $u = 1/r$,

$$E = \frac{\ell^2}{2\mu}[(du/d\phi)^2 + u^2] - \Gamma u$$

Try a solution $u = A(1 - \epsilon\cos\phi)$ and have

$$A = \mu\Gamma/\ell^2;\ \epsilon^2 = 1 + (2\ell^2 E/\mu\Gamma^2)$$

This will cover bounded (elliptical) and unbounded (hyperbolic) orbits of Kepler $\Gamma > 0$ and unbounded (hyperbolic) orbits of Coulomb $\Gamma < 0$. It works even for $\epsilon > 1$, although then the range of admissible angles ϕ is limited $(r > 0)$.

9. Another calculation from the energy [M. Born, "Atomic Physics"], using $u = 1/r$,

$$\dot{r} = (dr/d\phi)\dot{\phi} = (\ell/\mu r^2)(dr/d\phi) = -(\ell/\mu)du/d\phi$$

$$E = \frac{\ell^2}{2\mu}[(du/d\phi)^2 + u^2] - \frac{\Gamma}{r} = \text{constant}$$

$$dE/d\phi = 0 = (du/d\phi)[\frac{\ell^2}{\mu}(d^2u/d\phi^2 + u) - \Gamma]$$

$$\rightarrow u = \frac{\mu\Gamma}{\ell^2} + A\cos\phi = \frac{\mu\Gamma}{\ell^2}[1 - \epsilon\cos\phi]$$

10. Next: (1) There is a direct construction using the Runge-Lenz vector, an additional conserved quantity for the inverse-square-law force. (2) The virial theorem gives a "simple" approach to Kepler #3. (3) Relate the algebraic form to the analytical geometry for an ellipse ($\epsilon < 1$) or hyperbola ($\epsilon > 1$).

B. Runge-Lenz [Bernoulli, Laplace..]vector G3.9

1. H. Goldstein, Am. J. Phys. **43**, 735 (1975); **44**, 1123 (1976).

2. There is an additional conserved vector for the Kepler problem. Start with Newton #2 for a general central force

$$\dot{\mathbf{p}} = f(r)\mathbf{r}/r$$

3. Then use vector product identity to form:

$$(r/f)[\dot{\mathbf{p}} \times \mathbf{L}] = \mu\mathbf{r} \times [\mathbf{r} \times \dot{\mathbf{r}}] = \mu[\mathbf{r}(\mathbf{r} \cdot \dot{\mathbf{r}}) - \dot{\mathbf{r}}r^2]$$

and use $(\mathbf{r} \cdot \dot{\mathbf{r}}) = \frac{1}{2}d(\mathbf{r} \cdot \mathbf{r})/dt = \frac{1}{2}d(r^2)/dt = r\dot{r}$ to form

$$\dot{\mathbf{p}} \times \mathbf{L} = (\mu f r^2)[\frac{\dot{r}}{r^2}\mathbf{r} - \frac{\dot{\mathbf{r}}}{r}] = -(\mu f r^2)\frac{d}{dt}(\mathbf{r}/r)$$

4. The central force has $d\mathbf{L}/dt = 0$ and Kepler potential $f = -\Gamma/r^2$:

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = \mu\Gamma\frac{d}{dt}(\mathbf{r}/r)$$

5. Hence there is a constant vector $\mathbf{A}$:

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \mu\Gamma\frac{\mathbf{r}}{r}$$

6. Take the dot product of this equation with $\mathbf{L}$. Since $\mathbf{L} \cdot \mathbf{r} = 0$ and $\mathbf{L} \cdot (\mathbf{p} \times \mathbf{L}) = 0$, $\mathbf{L} \cdot \mathbf{A} = 0$. Hence both $\mathbf{A}$ and $\mathbf{r}$ are in a plane $\perp$ $\mathbf{L}$.

7. Take the dot product of $\mathbf{A}$ with $\mathbf{r}$ and define ϕ by $\mathbf{r} \cdot \mathbf{A} = Ar\cos\phi$:

$$Ar\cos\phi = \mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) - \mu\Gamma r = \mathbf{L}^2 - \mu\Gamma r = \ell^2 - \mu\Gamma r$$

$$\frac{1}{r} = \frac{\Gamma\mu}{\ell^2}[1 + \frac{A}{\mu\Gamma}\cos\phi] \equiv C[1 + \alpha\cos\phi];\ \Gamma = C\ell^2/\mu; \alpha = A/C\ell^2 = A/\mu\Gamma$$

8. Set the value of A by putting this in the equation for the energy, using

$$\dot{r} = -r^2\frac{d}{dt}\frac{1}{r} = r^2 C\alpha\dot{\phi}\sin\phi = (\ell/\mu)C\alpha\sin\phi$$

$$E = \frac{1}{2}\mu[(\ell/\mu)C\alpha\sin\phi]^2 - \Gamma C(1 + \alpha\cos\phi) + \frac{\ell^2}{2\mu}C^2(1 + \alpha\cos\phi)^2$$

$$\alpha^2 = 1 + \frac{2\mu E}{C^2\ell^2} = 1 + \frac{2\ell^2 E}{\mu\Gamma^2} \equiv \epsilon^2$$

9. QM application: E. Merzbacher, 2nd edn., Problem 16.11; S. Weinberg, "Lectures on Quantum Mechanics" (Cambridge, 2013).