Lecture 37: Four-vectors (26 Nov 14)

Homework for December 8 is available as Suppl14.pdf

0. 24 November homework

A. Review: Lorentz transformation

1. Frames S and S^* moving with relative velocity v along $\hat{x}$

$$ct^* = \gamma(ct - \beta x);\ x^* = \gamma(x - \beta ct)$$

$$y^* = y;\ \ z^* = z;\ \ \beta = v/c; \gamma = 1/\sqrt{1-\beta^2}$$

2. notation for 4-vector: $x_\mu, \mu = 0, 1, 2, 3$; $x_0 = ct$, $x_1 = x, x_2 = y, x_3 = z$

3. Velocity transform, "elementary method" with differentials dx and dt

$$u_x = \frac{u_x^* + v}{1 + (vu_x^*/c^2)}; u_y = \frac{1}{\gamma}\frac{u_y^*}{[1 + (vu_x^*/c^2)]}$$

4. Clocks and rods. Events $E_i = (ct_i, \mathbf{r}_i)$; * in moving (rest) frame.

   (a) Compare time interval for clock at rest (same position in S^*) and when observed in lab frame S (2 places): time dilation.

   $$t_2 - t_1 = \gamma([t_2^* - t_1^*] + (v/c^2)[x_2^* - x_1^*]) \rightarrow \Delta t = \gamma \Delta t^*$$

   "High energy muons live longer." Gedanken experiment: measure time by the distance traveled divided by speed of light. Bounce light from a mirror with path at right angles to the relative velocity. Then calculate distance traveled as viewed in rest frame and as viewed in "lab"

   (b) Compare length of rod L_0 in rest frame S^* and L_S when measured (same times) in lab frame.[FitzGerald, Lorentz 1892-1893]

   $$x_2^* - x_1^* = \gamma([x_2 - x_1] - v[t_2 - t_1]) \rightarrow L_0 = \gamma L_S$$

   (c) Relativity of simultaneity: events simultaneous (equal times) in one frame appear at different times in a moving frame – same calculation as in (a).

B. Doppler effect

1. Transformation for plane electromagnetic waves that satisfy the wave equation [the wave operator retains its form under Lorentz transformation to moving frame S^*]

$$[\nabla^2 - \frac{\partial^2}{\partial t^2}]u(\mathbf{r}, t) = 0$$

2. Symon, Sec. 13-6; Jackson Sec.11.3.D. Plane e-m waves with forms in S and S^* (the latter being the rest frame of the source).

$$A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \theta);\;\; A^*\cos(\mathbf{k}^*\cdot\mathbf{r}^* - \omega^* t^* + \theta^*)$$

   Argue that the phase should be invariant ("counting of crests is independent of frame," Jackson Sec. 11.2.A, Møller, Sec.3). Then plug the expressions for x^*, etc. and collect coefficients of $x, t, ...$:

$$k_x = \gamma[k_x^* + (\beta/c)\omega^*]; \omega = \gamma[\omega^* + c\beta k_x^*]$$

$$k_y = k_y^*;\;\; k_z = k_z^*$$

3. Notice an invariance under Lorentz transformation:

$$c^2\mathbf{k}^2 - \omega^2 = c^2\mathbf{k}^{*2} - \omega^{*2}$$

4. Let propagation in the rest frame be in $x^* - y^*$ plane at angle α^* to $\hat{x}^*$,

$$k_x^* = (\omega^*/c)\cos\alpha^*; k_y^* = (\omega^*/c)\sin\alpha^*$$

   Then the Doppler effect is (ω in lab, higher frequency when approaching, $\alpha^* < \pi/2$)

$$\omega = \gamma\omega^*[1 + \beta\cos\alpha^*]$$

   The limit $\gamma \to 1$ gives the classical Doppler effect; there is an additional relativistic effect [clock rate] even if the relative motion is at right angles to the line of sight.

5. Transform of angles (second form was the stellar aberration result)

$$\cos\alpha = k_x c/\omega = \frac{\cos\alpha^* + \beta}{1 + \beta\cos\alpha^*};\;\; \tan(\alpha/2) = \sqrt{\frac{1-\beta}{1+\beta}}\tan(\alpha^*/2)$$

6. Non-relativistic Doppler shift by the invariance of the phase of a plane wave under Galilean transformation $t = t'; \mathbf{r} = \mathbf{r'} + \mathbf{v}t$, $\mathbf{k} = k\hat{n}$, $\hat{n}$ is a unit vector, of length 1 in both frames, $\hat{n} \cdot \hat{n} = \hat{n}' \cdot \hat{n}' = 1$

$$\phi = \omega t - \mathbf{k} \cdot \mathbf{r} = \omega[t - \frac{\hat{n} \cdot \mathbf{r}}{c}] = \omega'[t' - \frac{\hat{n}' \cdot \mathbf{r}'}{c'}]$$

Substitute $t = t', \mathbf{r} = \mathbf{r'} + vt'$ and equate coefficients:

$$\omega' = \omega[1 - \frac{\hat{n} \cdot \mathbf{v}}{c}]; \ \frac{\omega'}{c'}\hat{n}' = \frac{\omega}{c}\hat{n}$$

$$\omega'/c' = \omega/c \Rightarrow c'/c = \omega'/\omega = [1 - \frac{\hat{n} \cdot \mathbf{v}}{c}]; \ \hat{n} = \hat{n}'$$

C. Space-time algebra

1. Symon Sec. 14-1 [an overall sign difference from Goldstein]

2. Space-time distance between events $E_1 = (\mathbf{r}_1, ct_1)$ and $E_2 = (\mathbf{r}_2, ct_2)$ is

$$S_{21} = (\mathbf{r}_1 - \mathbf{r}_2)^2 - c^2(t_1 - t_2)^2 = \sum_{\mu=0}^{3} g_\mu (x_{\mu 1} - x_{\mu 2})^2$$

with $g_0 = -1, g_1 = g_2 = g_3$ [other conventions use overall minus sign $g_0 = 1, g_1 = g_2 = g_3 = -1$]

3. Represent the Lorentz transformation as a matrix operation:

$$x^*_\mu = \sum_\nu a_{\mu\nu} x_\nu$$

Composite of two transformations is the matrix product (group property checked for parallel velocities, Poincaré).

4. The Lorentz transformation on x_μ preserves the space-time norm:

$$\sum_\mu g_\mu x^*_\mu x^*_\mu = \sum_\mu g_\mu \sum_{\nu\lambda} a_{\mu\nu} x_\nu a_{\mu\lambda} x_\lambda = \sum_\nu g_\nu x_\nu x_\nu$$

$$\Rightarrow \sum_\mu g_\mu a_{\mu\nu} a_{\mu\lambda} = g_\nu \delta_{\nu\lambda}$$

16 equations reduce to $4 + 6 = 10$ distinct equations (i.e., 10 constraints – 6 free parameters in $a_{\mu\nu}$).

5. Then one can check that the inverse transform is

$$a^{-1}_{\nu\mu} = g_\nu g_\mu a_{\mu\nu} \ (\to) \sum_\mu g_\lambda g_\mu a_{\mu\lambda} a_{\mu\nu} = g_\lambda g_\nu \delta_{\nu\lambda} = \delta_{\nu\lambda}$$

6. The defining property of a 4-vector is that its transform is

$$A^*_\mu = \sum_\nu a_{\mu\nu} A_\nu$$

   Sums of 4-vectors are 4-vectors, multiply by a 4-scalar $\to$ a 4-vector.

7. If A_μ, B_μ are 4-vectors then the "inner product" $(A_\mu, B_\mu) = \sum_\mu g_\mu A_\mu B_\mu$ is a 4-scalar (invariant). However the sign is not guaranteed.

8. The calculation of the Doppler effect used the requirement that $\mathbf{k}\cdot\mathbf{r} - \omega t$ is a Lorentz scalar = the inner product of the 4-vectors $\mathbf{k}, \omega/c$ and $\mathbf{r}, ct$.

9. Derivative calculus: start from

$$x^*_\mu = \sum_\nu a_{\mu\nu} x_\nu ; x_\nu = \sum_\mu a^{-1}_{\nu\mu} x^*_\mu$$

   and use chain rule derivatives:

$$\frac{\partial}{\partial x^*_\mu} = \sum_\nu \frac{\partial}{\partial x_\nu} \frac{\partial x_\nu}{\partial x^*_\mu} = \sum_\nu \frac{\partial}{\partial x_\nu} a^{-1}_{\nu\mu} = \sum_\nu g_\mu g_\nu a_{\mu\nu} \frac{\partial}{\partial x_\nu}$$

   and so to a 4-vector and a 4-scalar (the wave operator)

$$\Box_\mu = g_\mu \frac{\partial}{\partial x_\mu}; \ g_\mu \frac{\partial}{\partial x^*_\mu} = \sum_\nu a_{\mu\nu} g_\nu \frac{\partial}{\partial x_\nu}; \ (\Box_\mu, \Box_\mu) = \sum_\mu g_\mu \Box_\mu \Box_\mu = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

10. Summary: a 4-vector transforms as $A^*_\mu = \sum_\nu a_{\mu\nu} A_\nu$ Some examples $x_\mu = (ct, \mathbf{r})$, $k_\mu = (\omega/c, \mathbf{k})$, $j_\mu = (c\rho, \mathbf{j})$, $\Box_\mu = g_\mu \partial/\partial x_\mu$, $p_\mu = (E/c, \mathbf{p})$. The scalar product of two 4-vectors is a Lorentz invariant:

$$(A_\mu, B_\mu) \equiv \sum_\mu g_\mu A_\mu B_\mu$$

   and $(\Box_\mu, \Box_\mu)$ is the wave equation operator; $(\Box_\mu, j_\mu) = 0$ is the equation of continuity.

$$(\Box_\mu, \Box_\mu) = \sum_\mu g_\mu \Box_\mu \Box_\mu = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$