Central Force Motion and the Physics of Running

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Chaos and Complex Systems Seminar

April 25, 2006

Practical and Impractical Questions from Last Year's Seminar

1. Why do squirrels hop?
2. Why do vertebrates exhibit a heat/work ratio of about 4?
3. Why aren't there any macroscopic wheeled animals?
4. If we ever meet aliens, would we be able to beat them in a 100 m dash? A marathon?
5. What is the optimum up-and-down "bobble"?
6. Can we associate life with a particular temperature?
7. Is optimal cadence trainable?

Outline

1. Motivation
2. Experimental “facts”
3. Mathematical Toy Model
4. Behavior of Model
5. Implications for Runners

Experimental Observations

I. All successful runners at all distances have cadences of 180 steps per minute or higher (Daniels, *Daniels Running Formula* (1998), p. 80).
II. The resultant ground force during stance phase passes very near the runner's center of gravity at all times (Cross, Am. J. Phys 67 (1999) 304-309).
III. Leg swing is energetically cheap, stance is energetically costly (Ruina et al, J. Th. Bio 237 (2005) 170-192; Griffin et al. J. App. Phys. 95 (2003) 172-183).
IV. Energy cost is proportional to the time integral of muscle tension (McMahon, *Muscles, Reflexes, and Locomotion* (1984), p. 214).

The Runner’s Dilemma

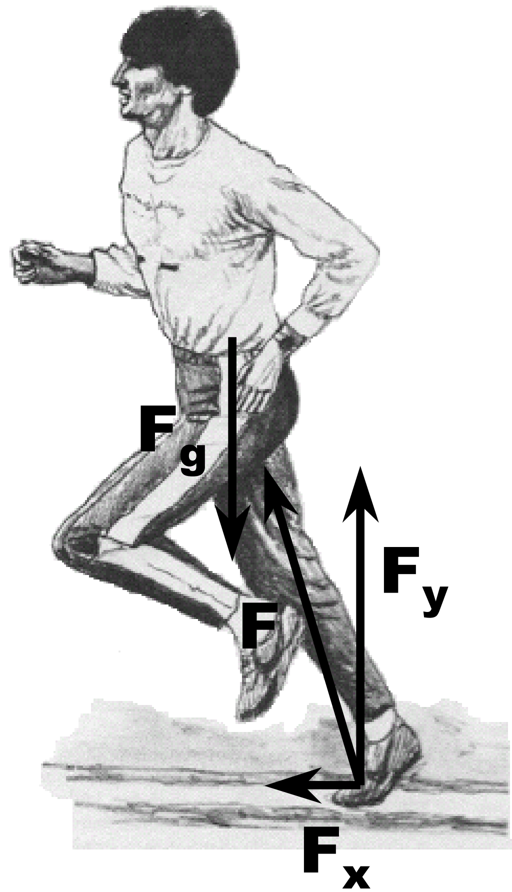
Don’t waste energy by bouncing up and down too much!

Don’t waste energy by staying on the ground too long!

It looks like there is an (non-zero) optimum up-and-down movement (aka “bobble”). Can we find the optimum?

Forces

F_g: force of gravity

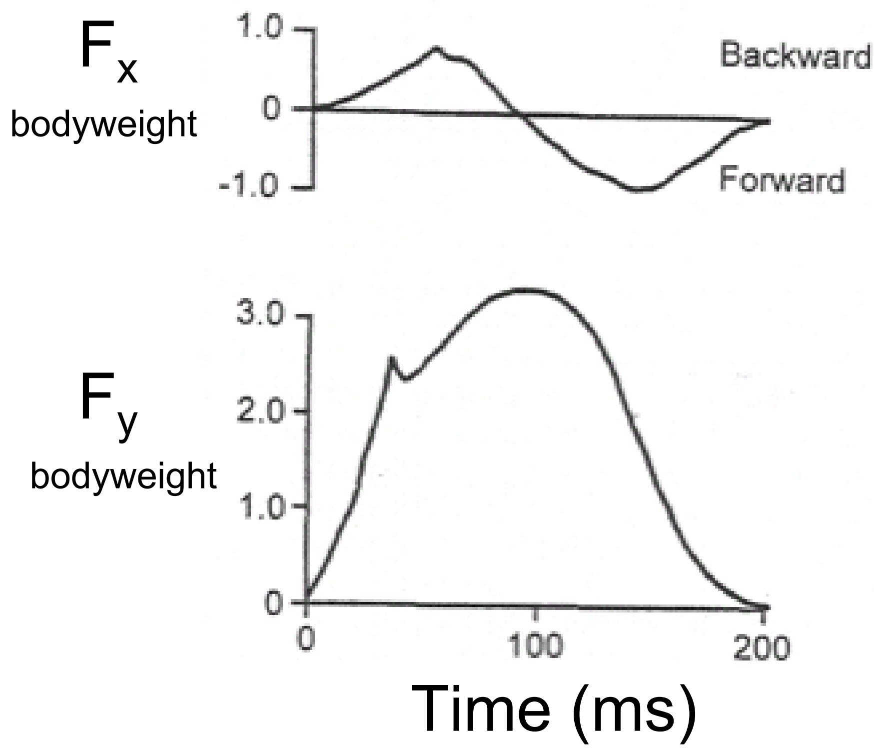


F: Ground Force

Components thereof:
F_x, F_y

Drawing of Seb Coe by Ron Daws

Ground Force--Data



Generalized force-time curves for two components of ground force during stance phase of running stride.

From Enoka, *Neuromechanics of Human Movement*, 3rd Edition, p. 79.

Ground Force--Theory

$$\int F_y dt = mgT$$

And there's nothing you can do about it!

$$\int F_x dt = 0$$

If you're moving at constant velocity.

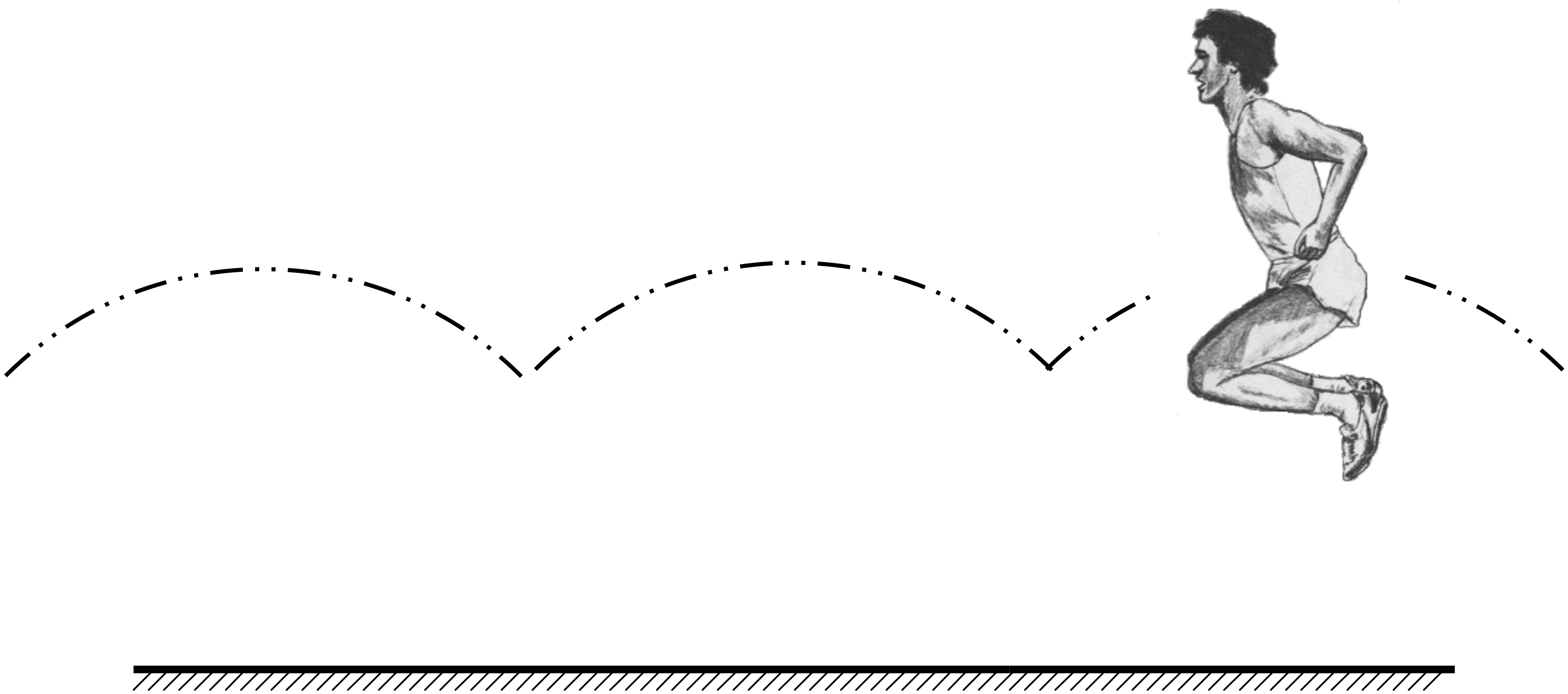
$$0 \leq \int \left| F_x \right| dt \leq \infty$$

This is up to you.

$$\int F dt = \int \sqrt{F_x^2 + F_y^2}\, dt$$

This is what you pay for.

The Runner's Dilemma--Part 1

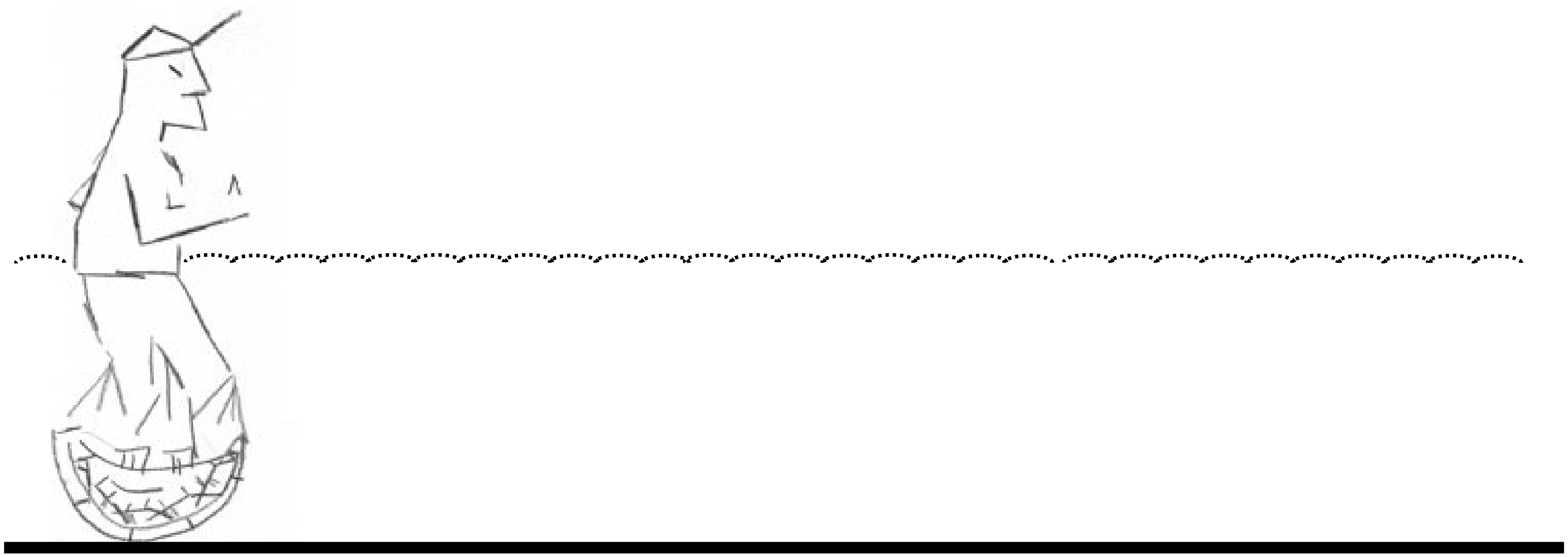


Infinitely short ground contact time leads to $\int |F_x| dt = 0$,

thus minimizing $\int F dt$.

However, this requires infinitely stiff legs.

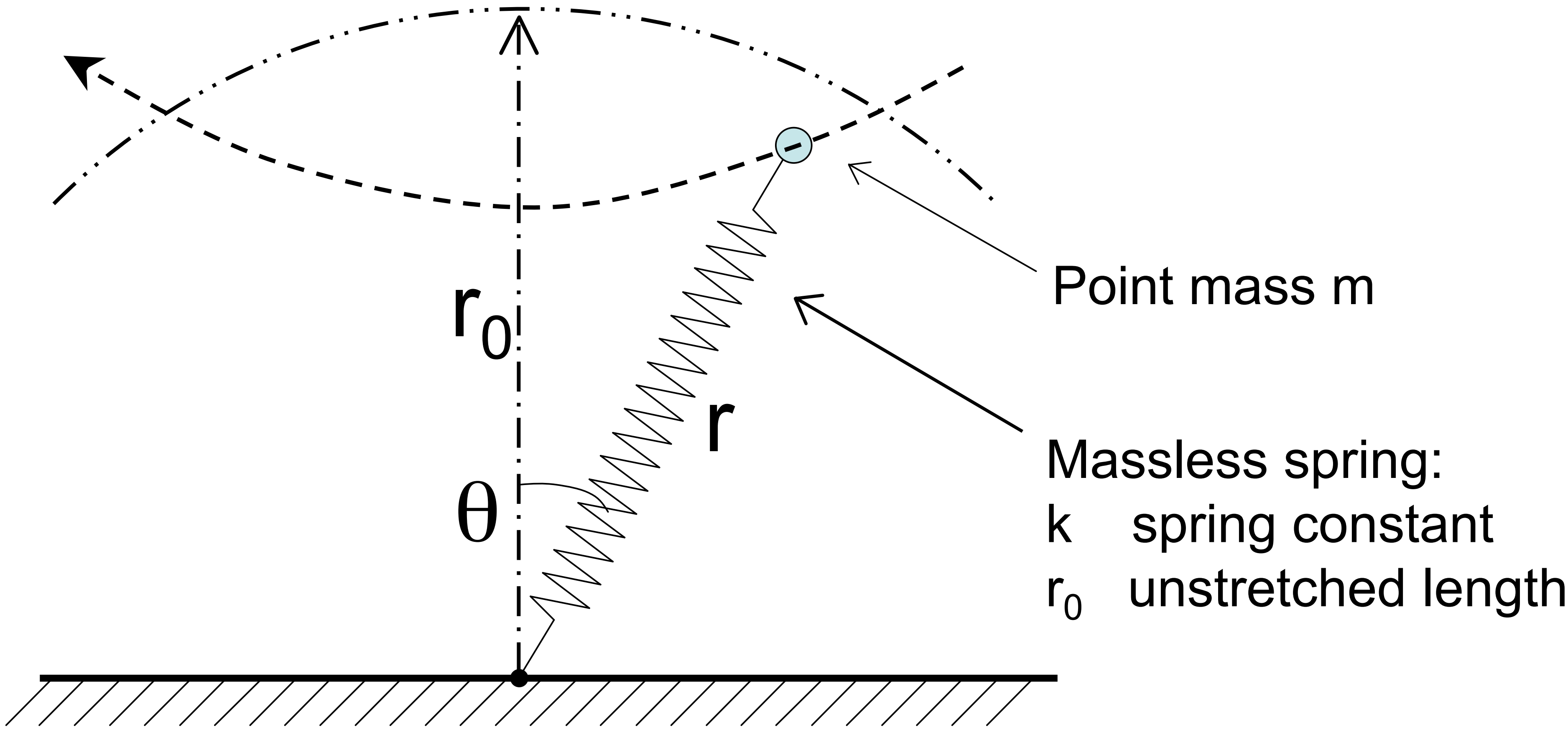
The Runner's Dilemma--Part 2



Running with an infinite cadence, leading to infinitely short steps, also produces $\int |F_x| dt = 0$.

However, the runner has now turned into a bicycle.

Toy Model



r_0
r
θ
Point mass m
Massless spring:
k spring constant
r_0 unstretched length

Toy Model

Mass follows "complicated" trajectory (to be calculated)

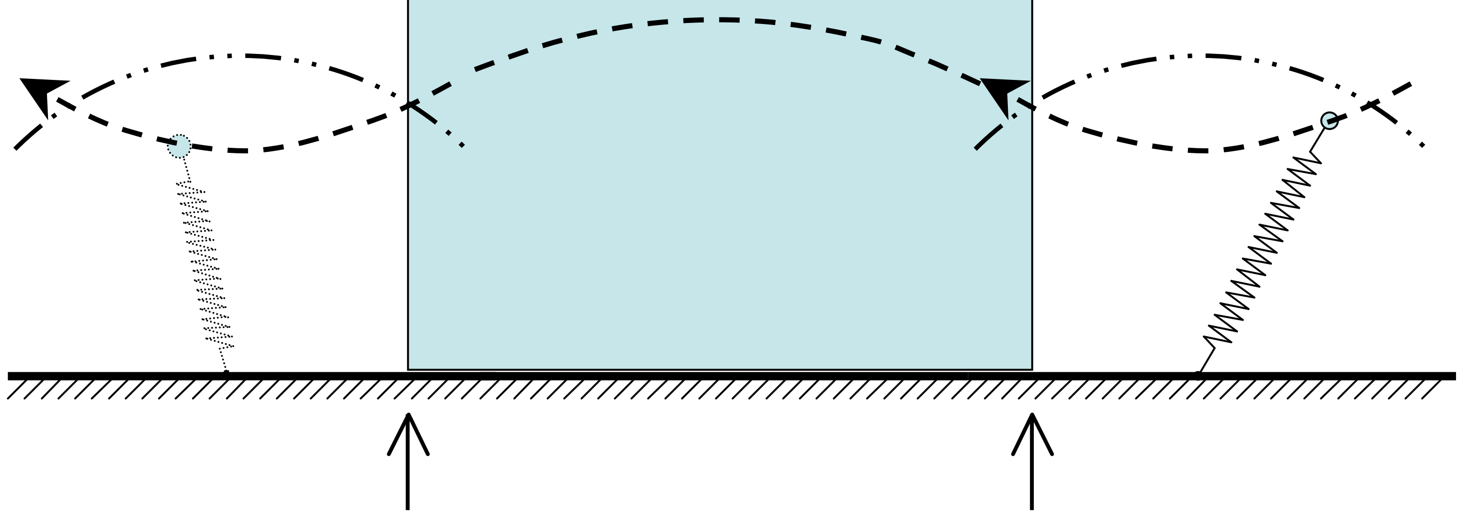
Energy cost $\propto \int F_r \, dt$

Mass follows ballistic trajectory (segment of parabola)

Energy cost=0

Mass follows "complicated" trajectory (to be calculated)

Energy cost $\propto \int F_r \, dt$



Spring contacts ground ("heel strike")

Spring reaches length r_0, leaves ground ("toe-off")

Springs

$$F_r = -k(r - r_0)$$

The force law

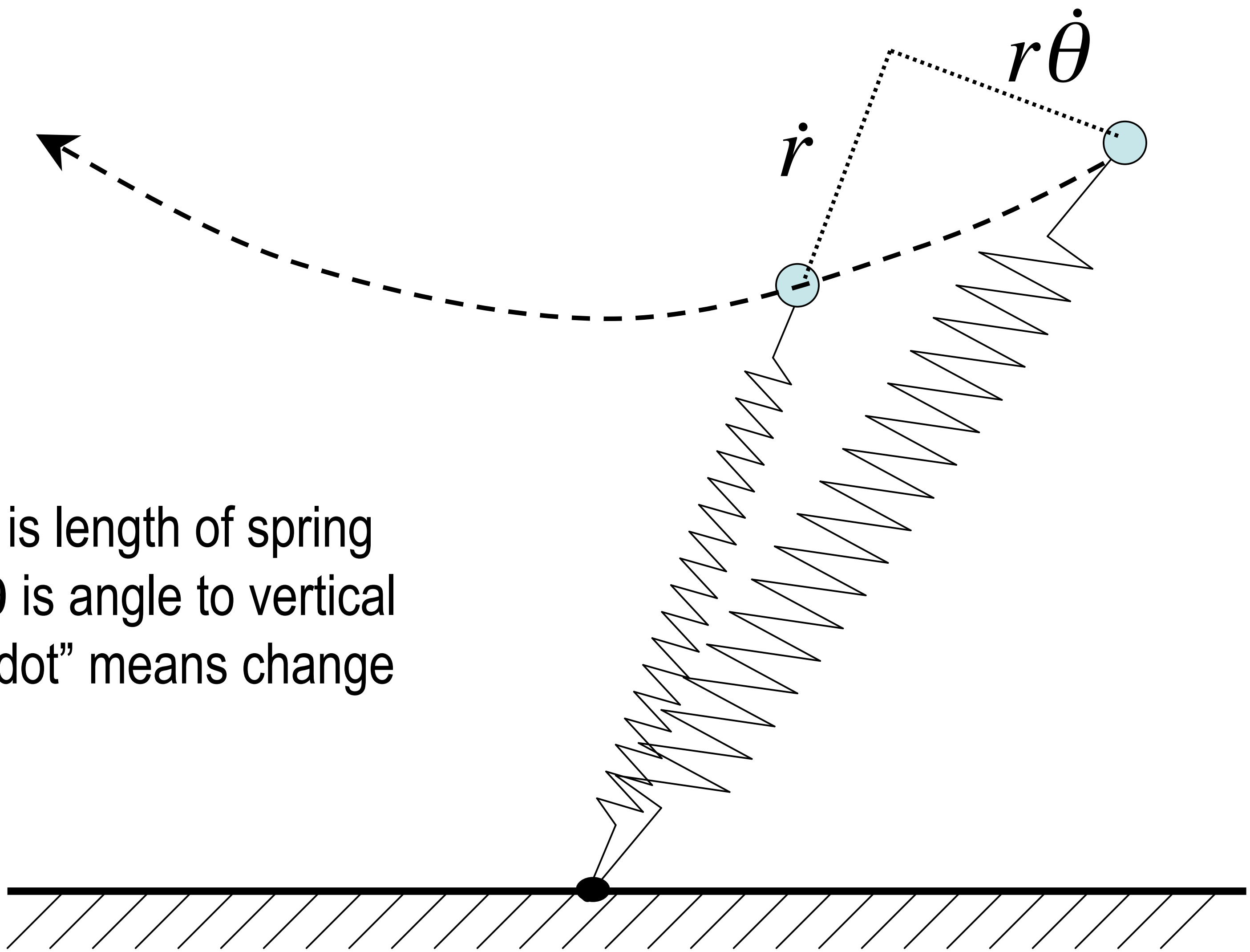
$$F_\theta = 0$$

Springs cannot exert force perpendicular to their length

$$E = \int \vec{F} \cdot d\vec{r} = \frac{1}{2}k(r - r_0)^2$$

A spring is an energy storage device

Change in position of mass from one instant to the next



r is length of spring
θ is angle to vertical
"dot" means change

Differential Equations

The expression of motion in the language of calculus

dr = change in length of spring

dt = change in time

$$\frac{dr}{dt} = \dot{r}$$

dθ = change in the angle the spring makes with the vertical

$$\frac{d\theta}{dt} = \dot{\theta}$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}k(r - r_0)^2$$

Energy

Energy = Kinetic Energy + Potential Energy

For Toy Model:

Energy = Energy associated with motion of mass parallel to spring + Energy associated with motion of mass perpendicular to spring + Energy stored in spring

$$E = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} k (r - r_0)^2$$

= constant

Central Force Motion

Force always directed towards one particular point (as in the toy model)

Force depends only on r, not on θ (as in toy model)

When these conditions hold, there is another constant of the motion in addition to E:

$$L = mr^2\dot{\theta}$$ Angular Momentum

This is central force motion.

Our Friend, Angular Momentum

Allows θ to be eliminated from differential equation:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{1}{2}k(r - r_0)^2$$

This can only be done for central force motion.

Algebra

Rewrite differential equation in a form that can be integrated.

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{1}{2}k(r - r_0)^2$$

Solve for $\dot{r}$:

$$\dot{r} = \left(\frac{2E}{m} - \frac{L^2}{m^2 r^2} - \frac{k}{m}(r - r_0)^2 \right)^{1/2} = \frac{dr}{dt}$$

Separate variables:

$$\frac{dr}{\left(\frac{2E}{m} - \frac{L^2}{m^2 r^2} - \frac{k}{m}(r - r_0)^2 \right)^{1/2}} = dt$$

More Algebra

Introduce dimensionless variables:

$$\rho = \frac{r}{r_0}, \;\; \tau = \sqrt{\frac{k}{m}}\; t$$

$$\frac{\rho\, d\rho}{\left(-\rho^4 + 2\rho^3 - (1 - \frac{2E}{kr_0^2})\,\rho^2 - \frac{L^2}{mkr_0^4}\right)^{1/2}} \;=\; d\tau$$

Now, integrate both sides…

3rd Annual University of Wisconsin

Integration Bee
First Prize $100

Open to all UW students

All integrals can be solved by techniques of Math 222

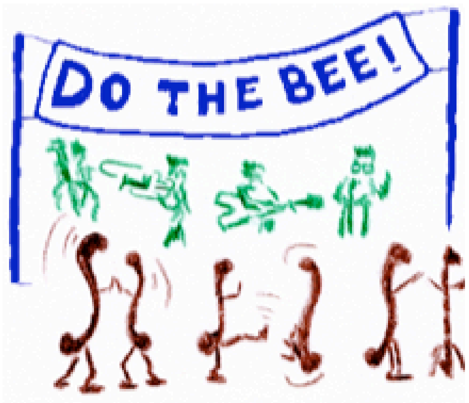
Qualifier (written): April 25 7:30-8:30 pm 1300 Sterling

Top 10 scores go to the finals.

Finals (live at the board): May 2 7:30 pm 1300 Sterling

Spectators welcome!

Dance Dance Integration!



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Roots: Algebra

Before integrating, we have to find the roots of the quartic

$$\rho^4 - 2\rho^3 + (1 - \frac{2E}{kr_0^2})\rho^2 + \frac{L^2}{mkr_0^4} = 0$$

Define:

$$a_3 = -2,\ a_2 = 1 - \frac{2E}{kr_0^2},\ a_1 = 0,\ a_0 = \frac{L^2}{mkr_0^4}$$

Then define:

$$p = a_2 - \frac{3}{8}a_3^2,\ q = a_1 - \frac{1}{2}a_2 a_3 + \frac{1}{8}a_3^3,$$

$$r = a_0 - \frac{1}{4}a_1 a_3 + \frac{1}{16}a_2 a_3^2 - \frac{3}{256}a_3^4$$

Roots: More Algebra

Then define:

$$b_0 = \frac{8}{3}rp - q^2 - \frac{2}{27}p^3, \; b_1 = -4r - \frac{p^2}{3}$$

Then define:

$$\Delta = \sqrt[3]{\frac{-b_0 \pm \sqrt{b_0^2 + \frac{4b_1^3}{27}}}{2}}$$

Then define:

$$u = \Delta - \frac{b_1}{3\Delta} + \frac{p}{3}$$

(make sure to pick the branch of the cube root that makes u real)

Roots: Roots

$$\rho_{out} = \frac{1}{2}\left(\sqrt{u-p} + \sqrt{u-p-\frac{2q}{\sqrt{u-p}}-2u-\frac{a_3}{2}}\right)$$

$$\rho_{in} = \frac{1}{2}\left(\sqrt{u-p} - \sqrt{u-p-\frac{2q}{\sqrt{u-p}}-2u-\frac{a_3}{2}}\right)$$

$$\rho_3 = \frac{1}{2}\left(-\sqrt{u-p} + \sqrt{u-p-\frac{2q}{\sqrt{u-p}}-2u-\frac{a_3}{2}}\right)$$

$$\rho_4 = \frac{1}{2}\left(-\sqrt{u-p} - \sqrt{u-p-\frac{2q}{\sqrt{u-p}}-2u-\frac{a_3}{2}}\right)$$

Algebra: Algebra

$$A_1 = \frac{1}{2}\left(\frac{-(u-p)^{3/2}+q}{\sqrt{q^2+2u(u-p)^2}}-1\right)$$

$$B_1 = \frac{1}{2}\left(\frac{(u-p)^{3/2}-q}{\sqrt{q^2+2u(u-p)^2}}-1\right)$$

$$A_1 + B_1 = -1$$

$$A_2 = \frac{1}{2}\left(\frac{-(u-p)^{3/2}-q}{\sqrt{q^2+2u(u-p)^2}}+1\right)$$

$$B_2 = \frac{1}{2}\left(\frac{(u-p)^{3/2}+q}{\sqrt{q^2+2u(u-p)^2}}+1\right)$$

$$A_2 + B_2 = 1$$

A_2>0 for realistic runners

Algebra, Algebra

Define one more pair of auxiliary quantities:

$$\alpha = \frac{1}{2}\left(\frac{q}{u-p} - \frac{a_3}{2} + \sqrt{\frac{q^2}{(u-p)^2} + 2u} \right)$$

$$\beta = \frac{1}{2}\left(\frac{q}{u-p} - \frac{a_3}{2} - \sqrt{\frac{q^2}{(u-p)^2} + 2u} \right)$$

$\beta < \rho < \alpha$ for the portion of the trajectory we are analyzing

Return to differential equations

Rewrite differential equation in terms of roots:

$$\frac{\rho\, d\rho}{\left((\rho_{out}-\rho)(\rho-\rho_{in})(\rho-\rho_3)(\rho-\rho_4)\right)^{1/2}} = d\tau$$

And then in terms of auxiliary quantites:

$$\frac{\rho \big/ (\rho-\beta)^2 \quad d\rho}{\left(\left(A_1\frac{(\rho-\alpha)^2}{(\rho-\beta)^2}+B_1\right)\left(A_2\frac{(\rho-\alpha)^2}{(\rho-\beta)^2}+B_2\right)\right)^{1/2}} = d\tau$$

Solution of equation of motion

Equation of motion can now be integrated, to give:

$$\tau = \tan^{-1}\left(\sqrt{\frac{A_2(\rho-\alpha)^2 + B_2(\rho-\beta)^2}{A_1(\rho-\alpha)^2 + B_1(\rho-\beta)^2}}\right) - \frac{\pi}{2}$$

$$+ \frac{1}{\sqrt{A_2B_1 - A_1B_2}}\left(\frac{\beta}{\alpha-\beta}F(\varphi,k) - A_1\Pi(\varphi,n,k)\right)$$

$F(\varphi,k)$ is elliptic integral of the first kind

$\Pi(\varphi,n,k)$ is elliptic integral of the third kind

$$\varphi = \sin^{-1}\sqrt{1 + \frac{A_1(\rho-\alpha)^2}{B_1(\rho-\beta)^2}},\ k = \sqrt{\frac{A_2B_1}{A_2B_1 - A_1B_2}},\ n = B_1$$

Definitions of φ, k, n follow *Numerical Recipes in Fortran*

Trajectory $\frac{d\rho}{d\theta}$

The trajectory of the mass can be found from:

$$\frac{d\rho}{d\theta} = \frac{d\rho/dt}{d\theta/dt}$$

Which leads to the differential equation:

$$\frac{d\rho}{\rho\left((\rho_{out} - \rho)(\rho - \rho_{in})(\rho - \rho_3)(\rho - \rho_4)\right)^{1/2}} = \frac{mr_0^2}{L}\sqrt{\frac{k}{m}}\,d\theta$$

Trajectory

$$\frac{mr_0^2}{L}\sqrt{\frac{k}{m}}\theta = \frac{\dfrac{\pi}{2} - \tan^{-1}\left(\sqrt{\dfrac{-A_1\alpha^2 - B_1\beta^2}{A_2\alpha^2 + B_2\beta^2}}\sqrt{\dfrac{A_2(\rho-\alpha)^2 + B_2(\rho-\beta)^2}{A_1(\rho-\alpha)^2 + B_1(\rho-\beta)^2}}\right)}{\sqrt{-A_1\alpha^2 - B_1\beta^2}\sqrt{A_2\alpha^2 + B_2\beta^2}}$$

$$+ \frac{1}{\beta\sqrt{A_2B_1 - A_1B_2}}\left(\frac{F(\varphi,k)}{\alpha-\beta} - \frac{A_1\alpha\,\Pi(\varphi,n,k)}{A_1\alpha^2 + B_1\beta^2}\right)$$

$$\varphi = \sin^{-1}\sqrt{1 + \frac{A_1(\rho-\alpha)^2}{B_1(\rho-\beta)^2}},\ \ k = \sqrt{\frac{A_2B_1}{A_2B_1 - A_1B_2}},\ \ n = \frac{-B_1\beta^2}{A_1\alpha^2 + B_1\beta^2}$$

Calculation of Energy Cost

If we had ρ as a function of t, we could calculate the ground force F(t) using:

$$F(t) = -k(r(t) - r_0) = -kr_0(\rho(t) - 1),$$

and then the energy cost \$ (in Joules) using:

$$\$ = \kappa \int F(t)\, dt$$

where κ is an unknown, anthropogenic constant.

Since we actually have t(ρ), not ρ(t), we need to do an integration by parts, which conveniently gives:

$$\$ = 2\kappa k \int_{\rho_{in}}^{1} t\, d\rho$$

An Approximation to Gravity

The force of gravity does not in general act along the line connecting the center of mass to the point of contact with the ground, so in reality the theory of central force motion does not apply.

So, we approximate.

In reality:

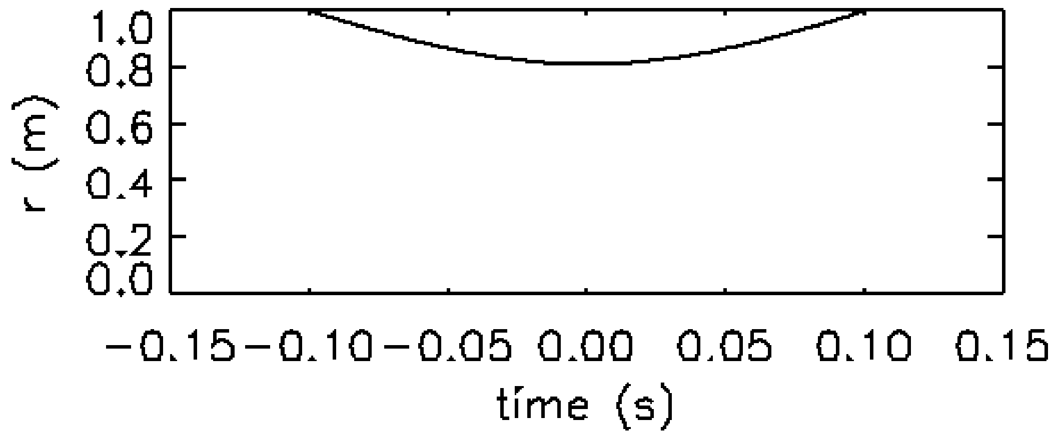
$$F_g = -mg\hat{y}$$
$$U_g = mgy$$

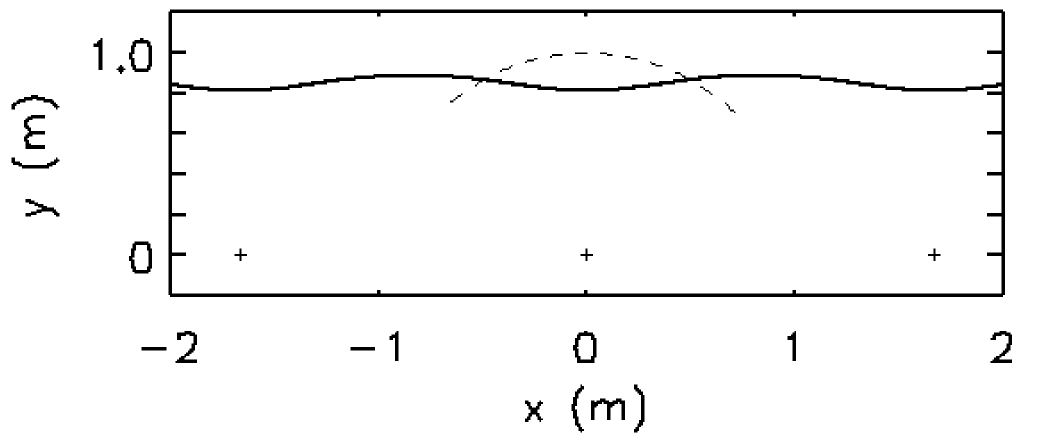
We approximate

$$F_g \approx -mg\hat{r}$$
$$U_g \approx mgr$$

We can then account for "gravity" by taking $a_3 = -2(1 - \frac{mg}{kr_0^2})$

Toy Model--Results





Inputs:

m= 67 kg

r_0= 1.0 m

k= 9090 N/m

L= 268.146 kg m^2/s

E= 1508.1 J

g= 9.8 m/s^2

Outputs:

v = 5.000 m/s

step length=1.6668 m

contact time=0.20001 s

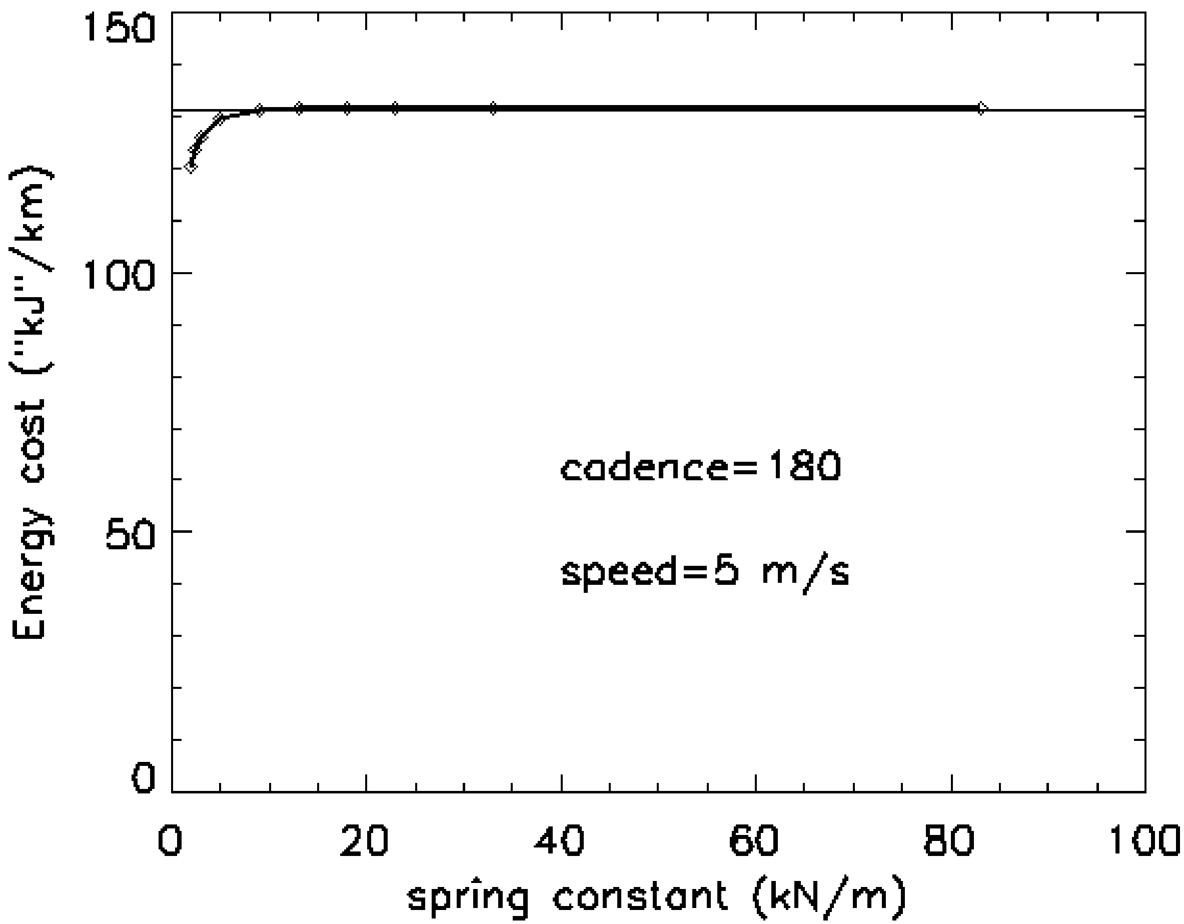
cadence =180.000 steps/min

$=218.86 J/step

=31.383 Cal/km

Toy Model--Results

Leg stiffness has little to do with energy cost

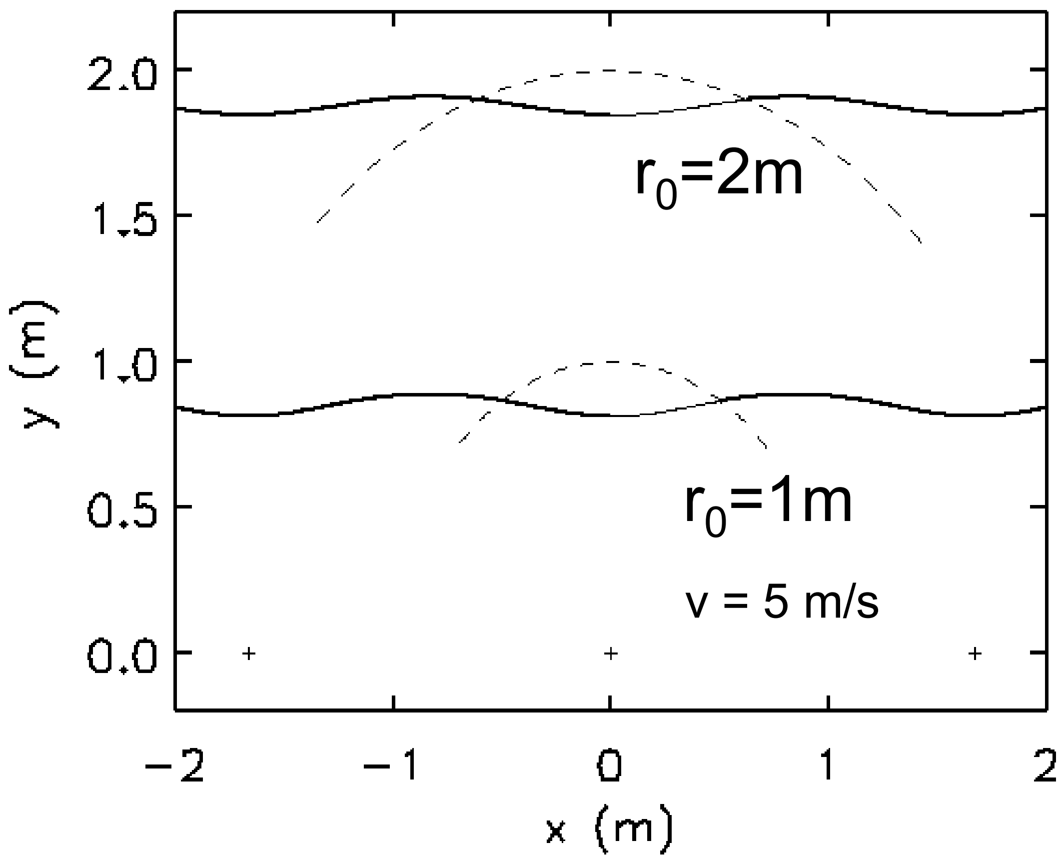


No obvious optimum "bobble"!

Toy Model--Results

Leg length has little effect on optimum energy cost

Comparison of trajectories of
Two different runners



For a given spring stiffness, taller runner will have longer contact time than smaller runner.

The Elephant Walk

Fast-moving elephants indulge in “Groucho running”

Hutchinson et al. Nature, 422 (2003), p. 493.

Future work

- Write computer program to numerically integrate exact equations of motion for toy model (ie account properly for gravity).

Conclusion

- Toy model does not provide solution to the runner’s dilemma. Optimum bobble is determined by the biodetails (eg configuration, behavior, control of neuromuscular system).