The Symmetric Linear Potential

The goal is to find analytical solutions to the TISE eigensystem for a potential of the form

$$V(x) = b|x| \qquad \left(x \in [-\infty, \infty]\right) \tag{1}$$

where b is a constant specific to the potential in question. The first thing to notice is that the potential is symmetric, which in this case means we can classify solutions in terms of parity. For $x > 0$, the TISE is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + bx\psi(x) = E\psi(x) \tag{2}$$

We change variables with the definitions[1]

$$x = as \qquad a = \left(\frac{\hbar^2}{mb}\right)^{1/3} \qquad W = \frac{\hbar^2}{ma^2} \qquad \varepsilon = \frac{E}{W} \qquad \xi = (s - \varepsilon) \tag{3}$$

and obtain a differential equation whose solutions are, in general, a linear combination of the Airy functions $Ai(\sqrt[3]{2}\xi)$ and $Bi(\sqrt[3]{2}\xi)$.

$$\psi''(\xi) = 2\xi\psi(\xi) \quad \Rightarrow \quad \psi(\xi) = C_1 Ai(\sqrt[3]{2}\xi) + C_2 Bi(\sqrt[3]{2}\xi) \tag{4}$$

However, Bi(ξ) diverges as $\xi \to \infty$. This violates the requirement that $\psi \to 0$ as $\xi \to \infty$, which does not make physical sense. We therefore restrict the solutions to the form

$$\psi_n(\xi) = C_n Ai(\sqrt[3]{2}\xi_n) \tag{5}$$

where C_n are normalization factors. Again, this is for $s > 0$.

At this point we consider the parity of our solutions. Even and odd eigenstates for $s > 0$ can be extended to give the solutions for $s < 0$ by

$$\psi^{even}(s) = \psi^{even}(-s) \qquad \psi^{odd}(s) = -\psi^{odd}(s) \tag{6}$$

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[1] Another option is to take $a = \left(\hbar^2/2mb\right)^{1/3}$ and $W = \hbar^2/2ma^2$. This leads to a more common form of the Airy equation, but includes a factor of $2^{-1/3}$ in the energy (9).

With these properties, it then stands to reason that we need the boundary conditions

$$\psi_k^{odd}(0) = Ai(-\sqrt[3]{2}\varepsilon_k) = 0 \qquad\qquad \frac{d\psi_k^{even}}{ds}(0) = Ai'(-\sqrt[3]{2}\varepsilon_k) = 0 \tag{7}$$

for our (non-normalized) solutions. Normalization is done numerically as

$$C_n = 2^{-1/3}\left[\int_{-\sqrt[3]{2}\varepsilon_n}^{\infty} Ai(\xi_n)^2 d\xi_n\right]^{-1/2} \tag{8}$$

The two equations in (7) then give the eigenvalues for odd and even parity states, respectively. Finally, we come to the conclusion that the bound state energies are given by

$$E_n = \varepsilon_n W = \varepsilon_n \left(\frac{\hbar^2 b^2}{m}\right)^{1/3} \tag{9}$$

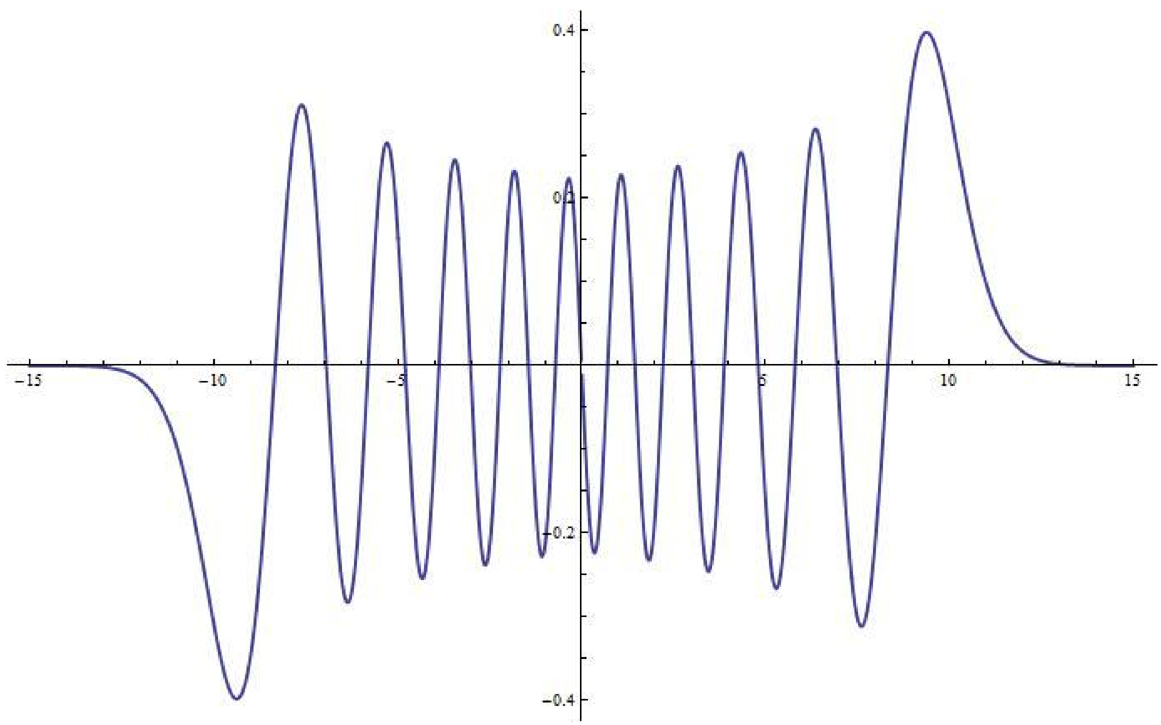

Fig 1: An example of the symmetric linear potential theoretical eigenfunctions (*n = 20*). Note that since ψ_{20} is an odd state, $\psi_{20}(0) = 0$. Observe also the decaying behavior of the wavefunction around the classical turning points $s_{\pm} = \pm\varepsilon_{20}$.