Supersymmetric Partner to Symmetric Well Problem

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We consider a potential $V(x) = \begin{cases} \frac{-mc^2}{2}, & \text{if } x \in (-\frac{\pi}{2}\frac{\hbar}{mc}, \frac{\pi}{2}\frac{\hbar}{mc}) \\ \infty, & \text{otherwise} \end{cases}$ If we let $W(s) = tan(s)$ and $A(s) = (\frac{\partial}{\partial s} + W(s))$, be a representation of an operator A in the position basis. Then, we have that $H_0 = \frac{1}{2}A^\dagger A$, where H_0 is the Hamiltonian for the particle in a symmetric well, as can be verified by expressing the operators in the position basis, with the substitution $s = \frac{xmc}{\hbar}$

The partner Hamiltonian is then $H_1 = \frac{1}{2}AA^\dagger = \frac{-1}{2}\frac{\partial^2}{\partial s^2} + sec(s)^2 - \frac{1}{2}$, in the position basis.

If we note that $AA^\dagger(A \mid n\rangle) = A(A^\dagger A) \mid n\rangle) = 2E_n A \mid n\rangle$, with $\mid n\rangle$ being the eigenvectors of H_0 and that the eigenvectors of H_0 are precisely (in the position basis) $\psi_n(s) = \begin{cases} \sqrt{\frac{2}{\pi}} cos(ns), & \text{if n is odd} \\ \sqrt{\frac{2}{\pi}} sin(ns), & \text{if n is even} \end{cases}$, which follows from the fact that $k = \sqrt{2E+1}$ but that boundary conditions dictate that $k = n$, we can immediately write down all eigenvectors (up to normalization) of H_1 as $A \mid n\rangle = (\frac{\partial}{\partial s} + Tan(s))(\psi_n(s))$.

We can explicitly determine this normalization, since, the partner eigenfunctions satisfy $A \mid n\rangle = c \mid n-1\rangle$, with c a complex number, so that $\|A \mid n\rangle\|^2 = \langle n \mid A^\dagger A \mid n\rangle = 2E_n =\mid c \mid^2$. Hence, we have that $\mid n_{partner}\rangle = \frac{1}{\sqrt{2E_{n-1}}} A \mid n-1\rangle$.